Czechoslovak Mathematical Journal, 62 (137) (2012), 663–672

k-TORSIONLESS MODULES WITH FINITE GORENSTEIN DIMENSION

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(Received March 7, 2011)

Abstract. Let R be a commutative Noetherian ring. It is shown that the finitely generated R-module M with finite Gorenstein dimension is reflexive if and only if $M_{\mathfrak p}$ is reflexive for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \leqslant 1$, and $\operatorname{G-dim}_{R_{\mathfrak p}}(M_{\mathfrak p}) \leqslant \operatorname{depth}(R_{\mathfrak p}) - 2$ for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \geqslant 2$. This gives a generalization of Serre and Samuel's results on reflexive modules over a regular local ring and a generalization of a recent result due to Belshoff. In addition, for $n \geqslant 2$ we give a characterization of n-Gorenstein rings via Gorenstein dimension of the dual of modules. Finally it is shown that every R-module has a k-torsionless cover provided R is a k-Gorenstein ring.

Keywords: torsionless module, reflexive module, Gorenstein dimension

MSC 2010: 13D05

1. Introduction

Let R be a commutative Noetherian ring with identity and let M be a finitely generated R-module. For an R-module M, the dual M^* of M is the R-module $\operatorname{Hom}_R(M, R)$. There is a natural evaluation map $\delta_M\colon M \to M^{**}$ and M is called reflexive provided δ_M is an isomorphism. It is clear that every finitely generated free module is reflexive. In 1958, Serre proved that every reflexive module is free provided R is a regular local ring of dimension at most 2. Later, Samuel extended this result by showing that over a regular local ring of dimension at most 3, an R-module M is reflexive if and only if $\operatorname{pd}_R M \leqslant 1$ and for every non-maximal prime ideal $\mathfrak p$, the localization $M_{\mathfrak p}$ is free over $R_{\mathfrak p}$. Recently, Belshoff considered an analogous question for modules over Gorenstein rings of low dimension. In particular, there is a notion of Gorenstein dimension for R-modules M, which satisfies the inequality

The research of S. Yassemi was in part supported by a grant from IPM No. 90130214.

$\text{G-dim}_R(M) \leqslant \text{pd}_R(M)$. This provides a natural setting for seeking extensions of the results of Serre and Samuel. In [3], Belshoff established the following:

(1) If R is a Gorenstein local ring of dimension at most 2, then every reflexive module M has $\text{G-dim}_R(M) = 0$.

(2) If R is a Gorenstein local ring of dimension 3, then an R-module M is reflexive if and only if $\text{G-dim}_R(M) \leqslant 1$ and the localization $M_{\mathfrak{p}}$ has Gorenstein dimension 0 over $R_{\mathfrak{p}}$ for every non-maximal prime ideal $\mathfrak{p}$.

In Section 2 we give a generalization of Belshoff's result (see Theorem 2.3). Recall that a ring R is called n-Gorenstein if the injective dimension of R is at most n. In Section 3, for $n \geqslant 2$ we give a characterization of n-Gorenstein rings via Gorenstein dimension of the dual of modules (see Theorem 3.4).

In Section 4 we bring a characterization of k-torsionless modules with finite Gorenstein dimension (see Theorem 4.5). Also this section concludes with a discussion of k-torsionless covers. Recall that for a class $\mathscr{X}$ of modules and a module M, an R-homomorphism $\varphi\colon X \to M$ is an $\mathscr{X}$-cover of M provided it is left universal among all homomorphisms $Y \to M$ with $Y \in \mathscr{X}$ and, further, any $X \to M$ factors through φ via an automorphism $X \to X$. In addition, we investigate necessary and sufficient conditions which lead the tensor product of k-torsionless modules to be k-torsionless.

2. Reflexive modules

This section contains some general remarks about reflexive modules with finite Gorenstein dimension. Indeed we present generalizations of the results of Belshoff [3], Serre [12] and Samuel [11].

First, we recall some necessary definitions which will be used in this section.

Definition 2.1. Let R be a ring and let M be an R-module. The dual of M is the module $\text{Hom}_R(M, R)$, which we usually denote by M^*, the bidual then is M^{**}, and anologous conventions apply to homomorphisms. The bilinear map $M \times M^* \longrightarrow R$, $(x, \varphi) \longmapsto \varphi(x)$, induces a natural homomorphism $\delta_M\colon M \longrightarrow M^{**}$. We say that M is torsionless if δ_M is injective, and that M is reflexive if δ_M is bijective.

In [1], Auslander and Bridger introduced the Gorenstein dimension, $\text{G-dim}_R(M)$, for every finitely generated R-module as follows:

Definition 2.2. A finitely generated R-module M is said to have *G*-dimension zero ($\text{G-dim}_R(M) = 0$) if and only if M satisfies the following three properties:

(i) M is reflexive,

(ii) $\text{Ext}^i_R(M, R) = 0$ for each $i \geqslant 1$,

(iii) $\text{Ext}^i_R(M^*, R) = 0$ for each $i \geqslant 1$.

Also we say that $\text{G-dim}_R(M) \leqslant n$ if there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow \ldots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $\text{G-dim}_R(G_i) = 0$ for each $i \geqslant 0$.

For every R-module M the inequality $\text{G-dim}_R(M) \leqslant \text{pd}_R(M)$ is proved by Auslander and Bridger in [1]. They showed that the equality holds if $\text{pd}_R(M)$ is finite. Furthermore, the Gorenstein dimension of an R-module is closely related to its depth, i.e., $\text{G-dim}_R(M) = \text{depth}(R) - \text{depth}_R(M)$. This equality is called the Auslander-Bridger formula.

The following theorem is special case of [4, Proposition 1.4.1] and gives a generalization of [3, Proposition 1.1, Proposition 1.7].

Theorem 2.3. *Let R be a ring and let M be an R-module with $\text{G-dim}_R(M) < \infty$. Then the following statements hold.*

(1) *M is torsionless if and only if*
   (i) *$M_\mathfrak{p}$ is torsionless for $\mathfrak{p} \in \text{Ass}(R)$, and*
   (ii) *$\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \text{depth}(R_\mathfrak{p}) - 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 1$.*

(2) *M is reflexive if and only if*
   (i) *$M_\mathfrak{p}$ is reflexive for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \leqslant 1$, and*
   (ii) *$\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \text{depth}(R_\mathfrak{p}) - 2$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 2$.*

Proof. (1) Assume that M is torsionless. By [4, Proposition 1.4.1], we must show that $\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \text{depth}(R_\mathfrak{p}) - 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 1$. Note that according to [4, Proposition 1.4.1], $\text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 1$. Now by the Auslander-Bridger formula we have

$$\text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \text{depth}(R_\mathfrak{p}) - \text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \text{depth}(R_\mathfrak{p}) - 1.$$

Conversely, by [4, Proposition 1.4.1], it suffices to show that $\text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 1$. By hypothesis and the Auslander-Bridger formula we have

$$\text{depth}(R_\mathfrak{p}) - \text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \text{G-dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leqslant \text{depth}(R_\mathfrak{p}) - 1,$$

therefore, $\text{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \geqslant 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_\mathfrak{p}) \geqslant 1$.

(2) is proved along the same lines as (1). □

The following two results which can be found in [3], [12] and [11], follow from Theorem 2.3.

Corollary 2.4. *Let* $(R,\mathfrak{m})$ *be a regular or Gorenstein ring with* $\dim(R)\leqslant 2$. *If* M *is a reflexive* R*-module, then* $\mathrm{pd}_R(M)=0$ *or* $G\text{-}\dim_R(M)=0$.

P r o o f. By [4, Theorem 2.2.7] or [6, Theorem 1.27] the projective dimension of M is finite or the G-dimension of M is finite, since R is regular or Gorenstein, respectively. By assumption $\mathrm{depth}(R_{\mathfrak{p}})\leqslant 2$ for all $\mathfrak{p}\in\mathrm{Spec}(R)$. If $\mathrm{depth}(R_{\mathfrak{p}})=2$, then by Theorem 2.3(2), $\mathrm{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$ ($\text{G-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$). Otherwise, $M_{\mathfrak{p}}$ is reflexive by Theorem 2.3(2) and by [4, Exercise 1.4.19] we have

$$\mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=\mathrm{depth}_{R_{\mathfrak{p}}}(\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}^{*},R_{\mathfrak{p}}))\geqslant\min\{2,\mathrm{depth}(R_{\mathfrak{p}})\}.$$

So $\mathrm{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$ ($\text{G-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$). □

Corollary 2.5. *Let* $(R,\mathfrak{m})$ *be a regular or Gorenstein ring of dimension* 3. *An* R*-module* M *is reflexive if and only if* $\mathrm{pd}_R(M)\leqslant 1$ *or* $G\text{-}\dim_R(M)\leqslant 1$ *and* $\mathrm{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$ *or* $G\text{-}\dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})=0$, *respectively, for every prime ideal* $\mathfrak{p}$ *distinct from* $\mathfrak{m}$.

P r o o f. It is proved along the same lines as Corollary 2.4. □

3. Characterization of n-Gorenstein rings

In this section, for $n\geqslant 2$ we give a characterization of n-Gorenstein rings via Gorenstein dimension of the dual of modules. Before that, we recall some definitions and properties. We follow standard notation and terminology from [7].

Definition 3.1. A Gorenstein ring with $\mathrm{id}_R(R)$ at most n is called n-Gorenstein.

Definition 3.2. A submodule T of an R-module N is said to be a pure submodule if $0\to A\otimes_R T\to A\otimes_R N$ is exact for all R-modules A, or equivalently, if $\mathrm{Hom}_R(A,N)\to\mathrm{Hom}_R(A,N/T)\to 0$ is exact for all finitely presented R-modules A.

An exact sequence $0\to T\to N\to N/T\to 0$ (or $0\to T\to N$) is said to be pure exact if T is a pure submodule of N. An R-module M is said to be pure injective if for every pure exact sequence $0\to T\to N$ of R-modules, the induced sequence $\mathrm{Hom}_R(N,M)\to\mathrm{Hom}_R(T,M)\to 0$ is exact.

The following lemma and theorem improve results due to Belshoff [3], where he studied the case $n=2$. Now we generalize them for $n\geqslant 2$.

Lemma 3.3. *Let N be an R-module and let $n \geqslant 2$ be an integer. Then $\mathrm{id}_R(N) \leqslant n$ provided that $\mathrm{Ext}_R^{n-1}(M^*, N) = 0$ for every finitely generated R-module M.*

P r o o f. Let

$$E^\bullet \colon\ 0 \to N \to E^0 \xrightarrow{\alpha^1} E^1 \xrightarrow{\alpha^2} E^2 \xrightarrow{\alpha^3} \dots$$

be an injective resolution of N. Since the Hom evaluation morphism

$$\theta_{MRE^\bullet} \colon\ M \otimes_R \mathrm{Hom}_R(R, E^\bullet) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), E^\bullet)$$

is an isomorphism of complexes (see [5]) and $\mathrm{Ext}_R^{n-1}(M^*, N) = 0$, so the $(n-1)$-st cohomology module of $M \otimes E^\bullet$ is zero. Therefore $0 \to M \otimes_R \mathrm{Im}(\alpha^n) \to M \otimes_R E^n$ is exact. This means that $\mathrm{Im}(\alpha^n)$ is a pure submodule of the injective module E^n, and now by [7, Lemma 9.1.5], $\mathrm{Im}(\alpha^n)$ is injective and we get the assertion. $\square$

It is straightforward that for a Gorenstein ring R, the following statements hold.

(i) If $\dim(R) = 0$, then all R-modules are reflexive,

(ii) If $\dim(R) = 1$, then all torsionless R-modules are reflexive.

Now we are ready to give a characterization of n-Gorenstein rings via Gorenstein dimension of the dual of modules.

Theorem 3.4. *For any integer $n \geqslant 2$, R is an n-Gorenstein ring if and only if for every finitely generated R-module M, $G\text{-}\dim_R(M^*) \leqslant n-2$.*

P r o o f. Let R be an n-Gorenstein ring and let M be a finitely generated R-module. By [13, Corollary 1.5], M^* is reflexive. It is straightforward that $(M^*_{\mathfrak{p}})$ is a reflexive $R_{\mathfrak{p}}$-module, for every prime ideal $\mathfrak{p}$ of R. By Theorem 2.3 we have $\mathrm{G\text{-}dim}_{R_{\mathfrak{p}}}(M^*_{\mathfrak{p}}) \leqslant n-2$, since $\dim(R_{\mathfrak{p}}) \leqslant n$. According to [6, Proposition 1.15] $\mathrm{G\text{-}dim}_R(M^*) \leqslant n-2$. Conversely, let M be a finitely generated R-module. By hypothesis $\mathrm{G\text{-}dim}_R(M^*)$ is finite, so we have

$$n-2 \geqslant \mathrm{G\text{-}dim}_R(M^*) = \sup\{i\colon\ \mathrm{Ext}_R^i(M^*, R) \neq 0\},$$

therefore $\mathrm{Ext}_R^{n-1}(M^*, R) = 0$. Now by Lemma 3.3 we conclude that $\mathrm{id}_R(R) \leqslant n$, so R is an n-Gorenstein ring. $\square$

In [7, Theorem 9.1.11], Enochs and Jenda showed that the property of being n-Gorenstein imposes nice conditions on the homological properties of modules over such rings. In the following we improve Enochs and Jenda's result.

Corollary 3.5. *Let $n \geqslant 2$ be an integer. For a ring R the following statements are equivalent.*

(1) *R is n-Gorenstein.*
(2) *$G\text{-}\dim_R(M^*) \leqslant n-2$ for all R-modules M.*
(3) *$\operatorname{id}_R(F) \leqslant n$ for all flat R-modules F.*
(4) *$\operatorname{id}_R(P) \leqslant n$ for all projective R-modules P.*
(5) *$\operatorname{fd}_R(E) \leqslant n$ for all injective R-modules E.*
(6) *$\operatorname{pd}_R(E) \leqslant n$ for all injective R-modules E.*

Proof. By Theorem 3.4, (1) and (2) are equivalent and by [7, Theorem 9.1.11], (1), (3), (4), (5) and (6) are equivalent. □

4. k-Torsionless Modules

In [10], Maşek defined the k-torsionless modules for $k \geqslant 0$. Indeed he gave a generalization of the torsionless and reflexive modules, i.e., torsionless modules are 1-torsionless and reflexive modules are 2-torsionless. The first result of this section is to give a generalization of [4, Proposition 1.4.1] for k-torsionless modules. As an application we show that the class of maximal Cohen-Macaulay modules and the class of k-torsionless modules are equivalent over Gorenstein local ring with dimension k. Finally, we show that every module over a k-Gorenstein ring has a k-torsionless cover.

Definition 4.1. Let M be a module, and let

$$(\pi)\colon\ P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \longrightarrow 0$$

be a projective presentation of M. The Auslander dual, $D(M)$, of M is defined as

$$D(M) = \operatorname{coker}(u^*\colon\ P_0^* \longrightarrow P_1^*),$$

in other words, dualizing (π) we get an exact sequence

$$(\pi^*)\colon\ 0 \longrightarrow M^* \xrightarrow{f^*} P_0^* \xrightarrow{u^*} P_1^* \longrightarrow D(M) \longrightarrow 0.$$

Clearly, $D(M)$ depends on which projective presentation (π) is used in the definition. In [10], Maşek proved the uniqueness of $D(M)$ up to projective equivalence. Moreover, Maşek proved that for an R-module M and natural R-homomorphism $\delta_M\colon\ M \to M^{**}$ we have

$$\ker(\delta_M) \cong \operatorname{Ext}^1_R(D(M), R), \quad \operatorname{coker}(\delta_M) \cong \operatorname{Ext}^2_R(D(M), R).$$

In addition, $\operatorname{Ext}_R^i(D(M),R) \cong \operatorname{Ext}_R^{i-2}(M^*,R)$, $\forall i \geqslant 3$. The following definition is from [10].

Definition 4.2. A module M is k-torsionless if $\operatorname{Ext}_R^i(D(M),R) = 0$, $\forall i = 1,\dots,k$. So 1-torsionless is torsionless, and 2-torsionless is reflexive. For $k \geqslant 3$, M is k-torsionless if M is reflexive and $\operatorname{Ext}_R^i(M^*,R)$, $\forall i = 1,\dots,k-2$.

Note that M is k-torsionless if and only if $M_{\mathfrak p}$ is k-torsionless over $R_{\mathfrak p}$ for all $\mathfrak p \in \operatorname{Spec}(R)$.

Definition 4.3. An R-module M possesses property $(\widetilde{S}_k)$ if

$$\operatorname{depth}_{R_{\mathfrak p}}(M_{\mathfrak p}) \geqslant \min\{k, \operatorname{depth}(R_{\mathfrak p})\}, \ \forall \mathfrak p \in \operatorname{Spec}(R).$$

Proposition 4.4. *Let M be an R-module such that G-$\dim_R(M) < \infty$ and let $k \geqslant 0$ be an integer. Then the following statements are equivalent.*

(i) *M is k-torsionless.*
(ii) *M possesses property $(\widetilde{S}_k)$.*

Proof. By [10, Theorem 42], (i) and (ii) are equivalent if G-dimension of M is locally finite. On the other hand, by [2, Corollary 6.3.4] "G-dimension of M is finite" is equivalent to "G-dimension of M is locally finite". □

The following theorem is one of the main results of this section. It gives a generalization of [4, Proposition 1.4.1] for k-torsionless modules.

Theorem 4.5. *Let R be a ring and let M be an R-module with G-$\dim_R(M) < \infty$. Then M is k-torsionless if and only if*

(i) *$M_{\mathfrak p}$ is k-torsionless for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \leqslant k-1$, and*
(ii) *$\operatorname{depth}_{R_{\mathfrak p}}(M_{\mathfrak p}) \geqslant k$ for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \geqslant k$.*

Furthermore, by the Auslander-Bridger formula, M is k-torsionless if and only if $M_{\mathfrak p}$ is k-torsionless for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \leqslant k-1$, and G-$\dim_{R_{\mathfrak p}}(M_{\mathfrak p}) \leqslant \operatorname{depth}(R_{\mathfrak p}) - k$ for $\mathfrak p \in \operatorname{Spec}(R)$ with $\operatorname{depth}(R_{\mathfrak p}) \geqslant k$.

Proof. Assume that M is a k-torsionless R-module, then (i) is straightforward and by Proposition 4.4, M possesses property $(\widetilde{S}_k)$. So (ii) holds.

Conversely, by Proposition 4.4, it suffices to show that M satisfies property $(\widetilde{S}_k)$, since G-$\dim_R(M) < \infty$. If $\operatorname{depth}(R_{\mathfrak p}) \geqslant k$, then by (ii), $\operatorname{depth}_{R_{\mathfrak p}}(M_{\mathfrak p}) \geqslant k$ and so $\operatorname{depth}_{R_{\mathfrak p}}(M_{\mathfrak p}) \geqslant \min\{k, \operatorname{depth}(R_{\mathfrak p})\}$. Otherwise by (i), $M_{\mathfrak p}$ is a k-torsionless $R_{\mathfrak p}$-module, hence $M_{\mathfrak p}$ possesses property $(\widetilde{S}_k)$. So M is k-torsionless. □

Corollary 4.6. *Let R be a Gorenstein local ring of dimension k and let M be a nonzero R-module. Then the following statements are equivalent.*

(i) *M is k-torsionless.*

(ii) *G-$\dim_R(M) = 0$.*

(iii) *M is maximal Cohen-Macaulay.*

Proof. (i) implies (ii) by Theorem 4.5.

(ii) $\Rightarrow$ (iii): Assume that G-$\dim_R(M) = 0$, so by the Auslander-Bridger formula the assertion holds.

(iii) $\Rightarrow$ (i): Suppose that M is a maximal Cohen-Macaulay R-module, so $M_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Supp}(M)$. Therefore M possesses property $(\widetilde{S}_k)$ and hence by Proposition 4.4, M is k-torsionless. □

In the following we study the covering properties of the class of k-torsionless modules. This result improves [3, Theorem 2.2].

Definition 4.7. Let $\mathscr{X}$ be the class of finitely generated k-torsionless R-modules. An $\mathscr{X}$-precover (it will be called a k-torsionless precover) of a finitely generated R-module M is defined to be an R-homomorphism $\varphi\colon C \to M$, for some $C \in \mathscr{X}$ such that for any R-homomorphism $f\colon D \to M$ where $D \in \mathscr{X}$, there is a homomorphism $g\colon D \to C$ such that $\varphi g = f$. An $\mathscr{X}$-precover $\varphi\colon C \to M$ is called an $\mathscr{X}$-cover (it will be called a k-torsionless cover) if whenever $g\colon C \to C$ is such that $\varphi g = f$, then g is an automorphism of C.

It is known that a projective precover of a module M always exists and when the ring R has the property that the direct limit of projective modules is projective, then M has a projective cover [7, Corollary 5.2.7]. Flat covers exist for all modules over any ring [7, Theorem 7.4.4]. In [3, Theorem 2.2], Belshoff proved that over a Gorenstein local ring of dimension at most 2, every finitely generated module has a reflexive cover. The next theorem gives a generalization of this result.

Theorem 4.8. *Let R be a k-Gorenstein ring, and let M be an R-module. Then M has a k-torsionless cover $C \to M$.*

Proof. By [7, Theorem 11.6.9], M has a Gorenstein projective cover $C \to M$ and C is finitely generated. It follows from Corollary 4.6 that $C \to M$ is the k-torsionless cover of M. □

In [8, Corollary 2.6] and [9], Huneke and Wiegand proved the following result: *Let R be a complete intersection ring and let M and N be nonzero R-modules such that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geqslant 1$. If $M \otimes_R N$ is maximal Cohen-Macaulay, then so are M and N.*

In the following, we provide necessary and sufficient conditions which lead the tensor product of k-torsionless modules to be k-torsionless.

Theorem 4.9. *Let R be a complete intersection ring with $\dim(R) = k$ and let M and N be nonzero R-modules such that $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geqslant 1$. Then $M \otimes_R N$ is k-torsionless if and only if M and N are k-torsionless.*

Proof. "$\Rightarrow$" Assume that $M \otimes_R N$ is k-torsionless. By Corollary 4.6, we get that $M \otimes_R N$ is maximal Cohen-Macaulay. Now by [8, Corollary 2.6], M and N are maximal Cohen-Macaulay, so by Corollary 4.6, M and N are k-torsionless.

"$\Leftarrow$" Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then M possesses property $(\widetilde{S}_k)$, since M is k-torsionless. So we have

$$\begin{aligned} \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}) &= \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) + \operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}/(\mathfrak{p}R_\mathfrak{p})N_\mathfrak{p}) \\ &\geqslant \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \\ &\geqslant \min\{k, \operatorname{depth}(R_\mathfrak{p})\}. \end{aligned}$$

Therefore $M \otimes_R N$ possesses property $(\widetilde{S}_k)$ and then by Proposition 4.4, the assertion is proved. □

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