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CLOSE COHOMOLOGOUS MORSE FORMS WITH COMPACT LEAVES

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Abstract. We study the topology of foliations of close cohomologous Morse forms (smooth closed 1-forms with non-degenerate singularities) on a smooth closed oriented manifold. We show that if a closed form has a compact leave γ, then any close cohomologous form has a compact leave close to γ. Then we prove that the set of Morse forms with compactifiable foliations (foliations with no locally dense leaves) is open in a cohomology class, and the number of homologically independent compact leaves does not decrease under small perturbation of the form; moreover, for generic forms (Morse forms with each singular leaf containing a unique singularity; the set of generic forms is dense in the space of closed 1-forms) this number is locally constant.

Keywords: Morse form foliation, compact leaf, cohomology class

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1. Introduction and statement of main results

Let M be a smooth closed oriented n-dimensional manifold and ω a Morse form on M, i.e., a smooth closed 1-form with Morse singularities—locally the differential of a Morse function (in the sequel, we will assume that all considered functions and forms are smooth). The set of its singularities $\operatorname{Sing}\omega$ is finite.

This form defines a foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$. Its leaves can be compactifiable—those that can be compactified by the form's singularities (including compact leaves)—and non-compactifiable. If all leaves of $\mathcal{F}_\omega$ are compactifiable, then $\mathcal{F}_\omega$ is called *compactifiable*; if it has no compactifiable leaves, then it is called *minimal.*

Morse forms are dense in the space of closed 1-forms in M supplied with the topology induced from C^∞. However, foliations of close Morse forms can have quite different topological structure: for example, a form with rational coefficients on a torus defines a compact foliation, but an arbitrary close form with irrational coefficients defines a winding, which is a minimal foliation.

Globally in the same cohomology class the foliation topology can also be quite different. For example, in any cohomology class having incommensurable periods there exists a Morse form with a minimal foliation [1], while in any cohomology class there are forms with compact leaves.

We show, however, that under some conditions the foliations of forms that are both cohomologous and close have similar topology. Namely, we show that some important classes are open in the space $F(\Omega)$ of closed 1-forms representing a class $\Omega \in H^1(M, \mathbb{R})$.

Non-compactifiable leaves are not stable under small perturbations of the form: for example, minimality of a foliation is not preserved for close cohomologous forms; there exist forms with minimal foliation that can be approximated by forms defining compactifiable foliation [9].

In contrast, compact leaves are stable under small perturbations of the form in its cohomology class. In particular, the set of Morse forms that define a compactifiable foliation is open in $F(\Omega)$ (Theorem 4.1).

The set of closed 1-forms that have a compact leaf is also open and non-empty in $F(\Omega)$ (Theorem 3.1). More precisely, a closed 1-form ω with a compact leaf γ has a neighborhood $\mathcal{O}(\omega) \subset F(\Omega)$ such that for any $\omega' \in \mathcal{O}(\omega)$ the foliation $\mathcal{F}_{\omega'}$ also has a compact leaf γ' that is close to γ and homologous to it, $[\gamma'] = [\gamma]$. The number $c(\omega)$ of homologically independent compact leaves does not decrease under small perturbations: $c(\omega') \geqslant c(\omega)$, while a strict inequality is possible (Example 3.1).

An important class of 1-forms are so-called generic forms: Morse forms with each singular leaf containing a unique singularity. This term, introduced in [3], is misleading because this property is not generic: while the set of generic forms is dense in $F(\Omega)$, it is—unlike the similarly defined class of functions—not necessarily open (Example 2.1). We show that the set of generic forms that define compactifiable foliation is open (though not necessarily dense) in $F(\Omega)$ and, unlike the non-generic case, the number of homologically independent compact leaves is locally constant: $c(\omega') = c(\omega)$ (Theorem 5.1).

The paper is organized as follows. In Section 2, we give necessary definitions and prove some useful facts. In Section 3, we study closed cohomologous forms that have a compact leaf. In Section 4, we consider Morse forms that define a compactifiable foliation. Finally, in Section 5, we study generic Morse forms that define a compactifiable foliation. In particular, we show that for such forms, the number of homologically independent compact leaves is a local invariant in the cohomology class.

2. Definitions and useful facts

Let M be a smooth closed oriented n-dimensional manifold.

2.1. Morse functions. A smooth function $f\colon M \to \mathbb{R}$ is called *Morse* if all its singularities (critical points) are non-degenerate. On a compact manifold its singular set $\operatorname{Sing} f = \{p \in M\,;\ df(p) = 0\}$ is finite because the singularities are isolated.

Proposition 2.1 ([6, II.6.2]). *The set of Morse functions is open and dense in the space $C^\infty(M, \mathbb{R})$ of all smooth functions on a given smooth manifold.*

Definition 2.1. We call a function f *generic* if it is a Morse function and all its critical values are distinct: $f(p) \neq f(q)$ for any $p, q \in \operatorname{Sing} f$, $p \neq q$.

In other words, each singular level of a generic function contains precisely one singularity. The term is motivated by the term *generic form* discussed in Section 2.4.

Definition 2.2 ([6, III.1.1]). Functions $f, f' \in C^\infty(M, \mathbb{R})$ are *equivalent* if there exist diffeomorphisms $g\colon M \to M$ and $h\colon \mathbb{R} \to \mathbb{R}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \mathbb{R} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ M & \xrightarrow{\ f'\ } & \mathbb{R} \end{array} \tag{2.1}$$

commutes. A function $f \in C^\infty(M, \mathbb{R})$ is called *stable* if there exists a neighborhood $\mathcal{O}(f) \subset C^\infty(M, \mathbb{R})$ such that each $f' \in \mathcal{O}(f)$ is equivalent to f.

Proposition 2.2 ([6, III.2.2]). *A function f on a compact manifold is stable iff it is generic.*

Proposition 2.3. *Let f be a generic function on M. Denote by $U_i = U(p_i) \subset M$ mutually disjoint neighborhoods of the singularities $\{p_i\} = \operatorname{Sing} f$. Then there exists a neighborhood $\mathcal{O}(f) \subset C^\infty(M, \mathbb{R})$ such that for any $f' \in \mathcal{O}(f)$ it holds $|\operatorname{Sing} f'| = |\operatorname{Sing} f|$ and $\operatorname{Sing} f' = \{p_i'\}$ with $p_i' \in U_i$.*

P r o o f. By Proposition 2.2, there exists a neighborhood $\mathcal{O}_1(f)$ with all functions equivalent to f. Consider $f' \in \mathcal{O}_1(f)$. The diagram (2.1) implies

$$df'(g(x)) \circ dg(x) = dh(f(x)) \circ df(x).$$

Since $dg(x)$ is an isomorphism, for any $p_i \in \operatorname{Sing} f$ it holds $df'(g(p_i)) = 0$, i.e., $g(\operatorname{Sing} f) \subseteq \operatorname{Sing} f'$. Since g is bijective, we obtain $|\operatorname{Sing} f| \leqslant |\operatorname{Sing} f'|$. Since equivalence of functions is symmetric, $|\operatorname{Sing} f| = |\operatorname{Sing} f'|$ and thus $\operatorname{Sing} f' = g\ (\operatorname{Sing} f)$.

Let $\operatorname{Diff}(M)$, the diffeomorphism group of M, be equipped with a topology. Denote by $\mathrm{id}_M \in \operatorname{Diff}(M)$ an identity element and by $\mathcal{O}(\mathrm{id}_M) \subset \operatorname{Diff}(M)$ its neighborhood such that for any $g \in \mathcal{O}(\mathrm{id}_M)$ it holds $g(p_i) \in U_i$ for all $p_i \in \operatorname{Sing} f$. Obviously, $\mathcal{O}_2(f) = \{f \circ g;\ g \in \mathcal{O}(\mathrm{id}_M)\}$ is a neighborhood of f in $C^\infty(M, \mathbb{R})$.

Since $p_i' = g(p_i)$, $\mathcal{O}(f) = \mathcal{O}_1(f) \cap \mathcal{O}_2(f)$ have the desired properties. □

Remark 2.1. Thus on a compact manifold M, the property for a function to be generic is a generic property: the set of such functions is open and dense in $C^\infty(M, \mathbb{R})$.

2.2. Morse forms. A *Morse form* ω is a closed 1-form with Morse singularities; locally it is the differential of a Morse function. Its singular set $\operatorname{Sing}\omega = \{p \in M;\ \omega(p) = 0\}$ consists of critical points of the corresponding Morse functions, which are non-degenerate. On a compact manifold, $\operatorname{Sing}\omega$ is finite.

Consider a singularity $p \in \operatorname{Sing}\omega$. Locally $\omega = df$; assume $f(p) = 0$. By the Morse lemma, there exists a neighborhood $\mathcal{O}(p)$ and smooth coordinates $x = (x^1, \dots, x^n)$ in $\mathcal{O}(p)$ such that $x(p) = 0$ and

$$f(x) = -(x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots + (x^n)^2,$$

where λ is the *index* of p, $\lambda = \operatorname{ind} p$.

If $\operatorname{ind} p = 0, n$, then locally the levels of f are spheres:

$$(x^1)^2 + \dots + (x^n)^2 = c,$$

where $c > 0$. Such a singularity p is called *spherical*.

If $\lambda = \operatorname{ind} p \neq 0, n$, then locally the levels of f are hyperboloids. The critical level is conic:

$$-(x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots + (x^n)^2 = 0.$$

Such singularity is called *conic*.

Since differential 1-forms on M are smooth sections of the cotangent bundle T^*M, the form ω can be considered as a smooth map $\omega\colon\ M \to T^*M$. All singularities $p \in \operatorname{Sing}\omega$ are non-degenerate, so the map ω is transversal to the zero section $M \subset T^*M$ at each point p [8]. Applying the Thom Transversality Theorem [6, II.4.12] to the map $\omega\colon\ M \to T^*M$, we obtain:

Lemma 2.1. *The set of Morse forms is open and dense in the space of closed 1-forms on M.*

2.3. Morse form foliation. A closed 1-form ω is integrable and thus it defines a foliation $\mathcal{F}_\omega$ on the set $M \setminus \operatorname{Sing}\omega$. In particular, a *leaf* $\gamma \in \mathcal{F}_\omega$ is a pathwise-connected integral surface of the distribution $\{\omega = 0\}$.

Remark 2.2. Two points $p, q \in M \setminus \operatorname{Sing}\omega$ belong to the same *leaf* iff there exists a path $\alpha\colon [0,1] \to M \setminus \operatorname{Sing}\omega$ connecting them such that $\omega(\dot\alpha(t)) = 0$ for all $t \in [0,1]$.

A leaf $\gamma \in \mathcal{F}_\omega$ is called *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact; otherwise it is called *non-compactifiable.* Note that a compact leaf is compactifiable; there is a finite number of non-compact compactifiable leaves. A foliation is called *compactifiable* if all its leaves are compactifiable.

The notion of foliation defined on $M \setminus \operatorname{Sing}\omega$ can be extended to the whole M to define so-called singular foliation. The definition is based on Remark 2.2:

Definition 2.3 ([2, 9.1])**.** A *singular foliation* $\overline{\mathcal{F}}_\omega$ is a decomposition of M into *leaves:* two points $p, q \in M$ belong to the same *leaf* iff there exists a path $\alpha\colon [0,1] \to M$ connecting them such that $\omega(\dot\alpha(t)) = 0$ for all $t \in [0,1]$.

By definition, a singular foliation $\overline{\mathcal{F}}_\omega$ has two types of leaves:

- ▷ Leaves that do not contain a singularity—so-called *non-singular leaves*; they are also leaves of $\mathcal{F}_\omega$.
- ▷ Leaves that contain a singularity; they are called *singular leaves* (hence the term *singular foliation*). While a spherical singularity itself is a singular leaf, a conic singularity is adjacent to at most four leaves of $\mathcal{F}_\omega$. Thus a leaf containing $p \in \operatorname{Sing}\omega$, $\operatorname{ind} p \neq 0, n$, consists of a finite (non-zero) number of leaves of $\mathcal{F}_\omega$ and some singularities.

Since $\operatorname{Sing}\omega$ is finite, there is a finite number of singular leaves—thus the "majority" of leaves of $\mathcal{F}_\omega$ and $\overline{\mathcal{F}}_\omega$ coincide.

Each compact leaf of $\mathcal{F}_\omega$ is a leaf of $\overline{\mathcal{F}}_\omega$. Each non-compact compactifiable leaf of $\mathcal{F}_\omega$ belongs to some singular leaf. For a compactifiable foliation, all leaves of $\overline{\mathcal{F}}_\omega$ are compact; singular leaves coincide with connected components of the union of non-compact leaves and singularities.

Definition 2.4 ([11])**.** A *regular neighborhood* U of $X \subset M$ is a locally flat, compact submanifold of M, which is a topological neighborhood of X such that the inclusion $X \hookrightarrow U$ is a simple homotopy equivalence and X is a strong deformation retract of U.

Since a compact singular leaf is a subcomplex of M viewed as a finite CW-complex, it has a regular neighborhood [7].

Let us prove the following auxiliary lemma:

Lemma 2.2. *Let F be a Morse function and $V = F^{-1}[a,b] \subseteq M$. If $\operatorname{Sing} F \cap \operatorname{Int} V = \{p\}$ then*

$$H_{n-1}(V) = i_* H_{n-1}(\partial V),$$

where $i\colon \partial V \hookrightarrow V$ is the inclusion map.

Proof. We only need to show $H_{n-1}(V) \subseteq i_* H_{n-1}(\partial V)$, since the converse is obvious.

Assume for simplicity that $F(p) = 0$ and $[a,b] = [-\varepsilon, \varepsilon]$. Denote by $\lambda = \operatorname{ind} p$ the index of the singularity. If $\lambda = 0, n$ then the result is trivial. Assume $\lambda \neq 0, n$.

Consider $V_- = F^{-1}[-\varepsilon, -\frac{1}{2}\varepsilon]$; see Figure 1. Since $V_- \cap \operatorname{Sing} F = \emptyset$, we have $V_- = F^{-1}(-\varepsilon) \times I$; therefore

$$H_{n-1}(V_-) = H_{n-1}(F^{-1}(-\varepsilon)), \tag{2.2}$$

where $F^{-1}(-\varepsilon) \subset \partial V$.

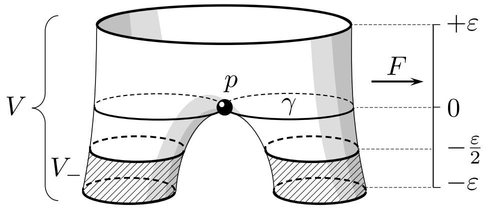

Figure 1. Regular neighborhood of a singular leaf γ

By the Morse theory, $V = V_- \cup_\varphi H^\lambda$, where $H^\lambda = D^\lambda \times D^{n-\lambda}$ is a handle and $\varphi\colon S^{\lambda-1} \times D^{n-\lambda} \to \partial V_-$ is a smooth embedding of the handle boundary. Consider the Mayer-Vietoris sequence for $V = V_- \cup_\varphi H^\lambda$. Since $H_{n-1}(H^\lambda) = 0$, we have the exact sequence:

(2.3)

$$\to H_{n-1}(S^{\lambda-1} \times D^{n-\lambda}) \to H_{n-1}(V_-) \to H_{n-1}(V) \to H_{n-2}(S^{\lambda-1} \times D^{n-\lambda}) \to .$$

Let $n \geqslant 3$. We assume that $\lambda \neq n-1$; otherwise we can consider the function $-F$, which defines the same V and has $\lambda = 1$. By the Künneth theorem, we have $H_{n-1}(S^{\lambda-1} \times D^{n-\lambda}) = 0$ and $H_{n-2}(S^{\lambda-1} \times D^{n-\lambda}) = 0$. So (2.3) implies that $H_{n-1}(V) = H_{n-1}(V_-)$. Since $F^{-1}(-\varepsilon) \subset \partial V$, (2.2) gives $H_{n-1}(V) \subseteq i_* H_{n-1}(\partial V)$.

Let now $n = 2$. In this case $\lambda = 1$ and (2.3) becomes

$$0 \to H_1(V_-) \to H_1(V) \xrightarrow{\partial_*} H_0(S^0 \times D^1) \to H_0(V_-) \oplus H_0(H^1) \to H_0(V) \to 0. \tag{2.4}$$

Since $\lambda = 1$, one of the levels $F^{-1}(-\varepsilon)$ or $F^{-1}(\varepsilon)$ has two connected components. We assume that it is $F^{-1}(-\varepsilon)$; otherwise we can consider the function $-F$. Then $H_0(V_-) = \mathbb{Z} \oplus \mathbb{Z}$ and (2.4) becomes

$$0 \to H_1(V_-) \to H_1(V) \xrightarrow{\partial_*} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0.$$

The sequence is exact, so $\operatorname{im} \partial_* = 0$, which again implies $H_1(V) = H_1(V_-)$, and (2.2) gives the result. □

2.4. Generic forms. Consider an important class of Morse forms, analogous to that of generic functions:

Definition 2.5 ([2, Definition 9.1])**.** A Morse form is called *generic* if each its singular leaf contains precisely one singularity.

The set of all generic forms is dense in the space of closed 1-forms (cf. Lemma 2.1):

Proposition 2.4 ([2, Lemma 9.2])**.** *Given a Morse form ω, there exists an arbitrarily small perturbation ω' in the same cohomology class that is generic.*

Remark 2.3. The term *generic form* introduced in [3] is misleading: generally this is not a *generic* property, i.e., a property that holds on a dense open set. The following example shows that the set of generic forms is not necessarily open.

Example 2.1. On a 2-torus, consider a form ω defining an irrational winding with two local perturbations with centers; see Figure 2. The form on the left is generic: the conic singularities p and q lie on different leaves. However, the leaf $\gamma \ni p$ is dense near q, so moving q slightly places it on γ: the form on the right is not generic. The two forms are cohomologous because both forms are cohomologous to the non-perturbed winding.

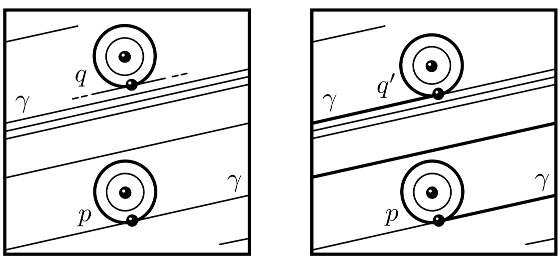

Figure 2. Left: a generic form on a torus T^2. Right: a non-generic form close to it.

The property of being generic is a generic property for functions (Remark 2.1) but not for forms (Remark 2.3). Example 2.1 differs from the case of functions in the existence of non-compactifiable leaves. In Section 5 we will show that without such leaves, i.e., when the foliation is compactifiable, the properties of generic forms are much closer to those of generic functions.

3. Closed forms that have a compact leaf

Any compact leaf has a cylindrical neighborhood consisting of leaves that are diffeomorphic and homotopically equivalent to it:

Lemma 3.1. *Let ω be a closed 1-form and $\gamma \in \mathcal{F}_\omega$ a compact leaf. Then for some neighborhood $U(\gamma)$ there exists a diffeomorphism*

$$\theta\colon\ \gamma \times (-\varepsilon, \varepsilon) \to U(\gamma)$$

such that $\theta(\gamma, t) = \gamma_t \in \mathcal{F}_\omega$ for any $t \in (-\varepsilon, \varepsilon)$.

P r o o f. Since ω is closed, its compact leaf γ has no holonomy. So by the Reeb local stability theorem, there exists a neighborhood $U = U(\gamma)$ saturated in $\mathcal{F}_\omega$, i.e. U consists of whole leaves that are compact. Note that $U \cap \operatorname{Sing}\omega = \emptyset$.

Since $\omega|_\gamma = 0$, in a regular neighborhood $V \supset U$ the form ω is exact, $\omega = dF$; so leaves of $\mathcal{F}_\omega$ are levels of $F\colon\ V \to \mathbb{R}$; assume $F|_\gamma = 0$. Then for some $\varepsilon > 0$ we have

$$U = \bigcup_{t\in(-\varepsilon,\varepsilon)} F^{-1}(t) = F^{-1}(-\varepsilon, \varepsilon). \tag{3.1}$$

Now let us show that U is diffeomorphic to $\gamma \times (-\varepsilon, \varepsilon)$. Consider in U the vector field

$$\xi^i_x = \frac{1}{|\operatorname{grad} F_x|^2_g} g^{ij} \partial_j F_x,$$

where $x \in U$ and g is a positive Riemannian metric globally defined on M. The vector field generates a flow $\widetilde{\theta}\colon\ U\times(-\varepsilon,\varepsilon) \to M$ with $\widetilde{\theta}(x, 0) = x$. Since $\dot{F} = \langle \operatorname{grad} F, \xi\rangle \equiv 1$, we have

$$F(\widetilde{\theta}(x, t)) = F(x) + t. \tag{3.2}$$

Denote by $\theta = \widetilde{\theta}|_{\gamma\times(-\varepsilon,\varepsilon)}$ the restriction of the flow $\widetilde{\theta}$ on γ. Since $F|_\gamma = 0$, (3.2) implies

$$F(\theta(y, t)) = t, \tag{3.3}$$

for any $y \in \gamma$, i.e., the flow maps the leaf γ to levels $\gamma_t = F^{-1}(t)$. Thus $\theta(\gamma, t) = \gamma_t \in \mathcal{F}_\omega$. We obtain $\operatorname{im}\theta = \bigcup_{t\in(-\varepsilon,\varepsilon)} F^{-1}(t) = F^{-1}(-\varepsilon, \varepsilon)$; obviously, θ is a diffeomorphism. By (3.1),

$$\theta : \gamma \times (-\varepsilon, \varepsilon) \to U$$

is the desired diffeomorphism. $\square$

Denote by $H_\omega \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$; denote $c(\omega) = \operatorname{rk} H_\omega$. By [4], there exist $c(\omega)$ homologically independent compact leaves γ_i that generate H_ω, i.e.

$$H_\omega = \langle [\gamma_1], \ldots, [\gamma_{c(\omega)}] \rangle. \tag{3.4}$$

Note that $c(\omega) \leqslant b_1(M)$, the first Betti number. Recall that $F(\Omega)$ is the space of closed 1-forms representing a class $\Omega \in H^1(M, \mathbb{R})$.

Theorem 3.1. *For any $\Omega \in H^1(M, \mathbb{R})$ and any $c \geqslant 0$, the following sets are open in $F(\Omega)$:*

(i) *the set of all closed 1-forms that have a compact leaf (this set is non-empty);*
(ii) *the set of all closed 1-forms that have at least c homologically independent compact leaves.*

Proof. (i) Let ω be a closed 1-form; if $\mathcal{F}_\omega$ has no compact leaves, consider $\omega + dh$, where h is a small bump function. This function has a spheric singularity, so the foliation defined by $\omega + dh$ also has a spheric singularity enclosed by compact leaves.

Let $\mathcal{F}_\omega$ has a compact leaf γ. Consider the neighborhood $U = U(\gamma)$ constructed in Lemma 3.1; in this neighborhood $\omega = dF$ with $\gamma = \{F(x) = 0\}$ and, by (3.1), $U = F^{-1}(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Let $\omega' = \omega + df$, where f is small enough, $f \in \mathcal{O}(0) \subset C^\infty(M)$; in particular, $|f| < \varepsilon/2$. Obviously, $\omega' = d(F + f)$ in U. Denote

$$\gamma' = \{F(x') + f(x') = 0\};$$

it is a leaf of $\mathcal{F}_{\omega'}$. Since $|F(x')| = |f(x')| < \varepsilon/2$, then $\gamma' \subset U(\gamma)$.

By Lemma 3.1, there exists a diffeomorphism

$$\theta\colon\ \gamma \times (-\varepsilon, \varepsilon) \to U,$$

such that $\theta(\gamma, t) = F^{-1}(t)$. Consider the leaves $\gamma, \gamma' \subset U$ and their diffeomorphic preimages $\theta^{-1}(\gamma) = \gamma$ and $\theta^{-1}(\gamma')$ in $\gamma \times (-\varepsilon, \varepsilon)$. Obviously, $\gamma = \{(y, 0)\}$, where y is a local coordinate in γ. The surface $\theta^{-1}(\gamma')$ is defined by the equation $F(\theta(y,t)) + f(\theta(y,t)) = 0$, which by (3.3) rewrites as

$$t + f(\theta(y,t)) = 0. \tag{3.5}$$

Consider a point $(y,0) \in \gamma$. Since f is smooth and small enough, by the implicit function theorem, in some neighborhoods $\mathcal{O}(y) \subset \gamma$ and $(-\varepsilon_y, \varepsilon_y) \subset (-\varepsilon, \varepsilon)$ the equation (3.5) defines a unique function $t_y\colon \mathcal{O}(y) \to (-\varepsilon_y, \varepsilon_y)$.

This allows us to represent $\theta^{-1}(\gamma')$ locally as a graph of a function φ_y on γ. In particular, for any $y' \in \mathcal{O}(y)$ we have $(y',0) \in \gamma$ and

$$\varphi_y(y',0) = (y', t_y(y')) \in \theta^{-1}(\gamma').$$

Consider a cover $\gamma \subset \bigcup_{y\in\gamma} \mathcal{O}(y)$. By the construction, $t_{y_1}(y) \equiv t_{y_2}(y)$ for all $y \in \mathcal{O}(y_1) \cap \mathcal{O}(y_2) \neq \emptyset$. Since γ is compact, there exists a finite subcover $\{\mathcal{O}(y_i)\}$. Thus we can construct a global function $t\colon \gamma \to (-\varepsilon, \varepsilon)$ that defines a global function $\varphi\colon \gamma \to \gamma \times (-\varepsilon, \varepsilon)$. In particular, for any $y \in \gamma$ we have $\varphi(y,0) = (y, t(y)) \in \theta^{-1}(\gamma')$. We obtain that $\theta^{-1}(\gamma')$ is a graph of a function φ on γ, and therefore $\theta^{-1}(\gamma')$ and $\theta^{-1}(\gamma) = \gamma$ are homologous in the cylinder $\gamma \times (-\varepsilon, \varepsilon)$; since θ is diffeomorphism, γ and γ' are also homologous, $[\gamma] = [\gamma']$.

(ii) There exist $c(\omega)$ homologically independent compact leaves $\gamma_i \in \mathcal{F}_\omega$ [4]. For each γ_i we construct $U(\gamma_i)$ as above. Choosing ω' close enough to ω, we obtain the corresponding $\gamma_i' \subset U(\gamma_i)$ such that $[\gamma_i'] = [\gamma_i]$ for all i; thus $c(\omega') \geqslant c(\omega)$. $\square$

Note that $c(\omega') > c(\omega)$ is possible:

Example 3.1. Consider a foliation on 2-torus with $c(\omega) = 0$. Slightly deforming the form we obtain $c(\omega') = 1$; see Figure 3.

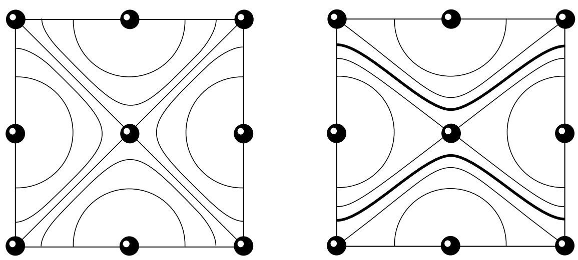

Figure 3. Left: $c(\omega) = 0$. Right: $c(\omega') = 1$; ω' is close to ω.

For a set $H \subseteq H_{n-1}(M)$, denote $H^{\ddagger} = \{z \in H_1(M);\ z \cdot H = 0\}$, where $\cdot$ is the cycle intersection. It is a subgroup; in addition, $H_1 \subseteq H_2$ implies $H_2^{\ddagger} \subseteq H_1^{\ddagger}$.

Let ω be a Morse form. Recall that a foliation is called compactifiable if all its leaves are closed in $M \setminus \operatorname{Sing}\omega$.

Proposition 4.1 ([4]). *Morse form foliation $\mathcal{F}_\omega$ is compactifiable iff*

$$H_\omega^{\ddagger} \subseteq \ker[\omega],$$

where $[\omega]\colon H_1(M) \to \mathbb{R}$ is the integration map.

Obviously, the integration map $[\omega]$ can be seen as a cohomology class $\Omega = [\omega]$; so we can define $\ker\Omega$ and $\operatorname{rk}\Omega$.

Theorem 4.1. *For any $\Omega \in H^1(M,\mathbb{R})$, the set of Morse forms that define a compactifiable foliation is open in $F(\Omega)$.*

Proof. Let ω be a Morse form that defines a compactifiable foliation $\mathcal{F}_\omega$, and $[\omega] = \Omega$. By Lemma 2.1, there exists a neighborhood $U(\omega) \subset F(\Omega)$ that consists only of Morse forms.

Consider the compact leaves $\{\gamma_1, \dots, \gamma_{c(\omega)}\}$ that generate H_ω, see (3.4). By Theorem 3.1, for each γ_i there exists a neighborhood $U_i(\omega) \subset F(\Omega)$ such that any $\omega' \in U_i(\omega)$ has a compact leaf γ_i' homologous to γ_i, $[\gamma_i'] = [\gamma_i]$. Denote

$$V(\omega) = U(\omega) \cap \bigcap_{i=1}^{c(\omega)} U_i(\omega).$$

Since $c(\omega)$ is finite, $V(\omega) \subset F(\Omega)$ is an open neighborhood.

By construction, any form $\omega' \in V(\omega)$ is Morse and has compact leaves $\gamma_i' \in \mathcal{F}_{\omega'}$ such that $[\gamma_i'] = [\gamma_i]$ for all $i = 1, \dots, c(\omega)$. Thus $H_\omega = \langle[\gamma_i']\rangle \subseteq H_{\omega'}$; in particular, $c(\omega) \leqslant c(\omega')$. Note that ω' can have compact leaves other than γ_i'; see Figure 3.

We obtain $H_{\omega'}^{\ddagger} \subseteq H_\omega^{\ddagger}$. By assumption, $\mathcal{F}_\omega$ is compactifiable, so Proposition 4.1 gives $H_\omega^{\ddagger} \subseteq \ker[\omega]$. Since ω' is cohomologous to ω, we have $\ker[\omega] = \ker[\omega']$. Thus $H_{\omega'}^{\ddagger} \subseteq \ker[\omega']$, and by Proposition 4.1, $\mathcal{F}_{\omega'}$ is compactifiable. □

Unlike the set of all forms that have a compact leaf, which is non-empty in any class $\Omega \in H^1(M,\mathbb{R})$, the set of all forms that define a compactifiable foliation can be empty:

Proposition 4.2 ([5]). *If $\operatorname{rk}\Omega > b_1'(M)$, where $b_1'(M)$ is the maximal rank of a free quotient group of the fundamental group $\pi_1(M)$ [10], then $F(\Omega)$ contains no Morse forms that define compactifiable foliation.*

Recall that a Morse form is called *generic* if each its singular leaf contains precisely one singularity. While generally the properties of generic forms and generic functions differ (Remarks 2.1 vs. 2.3), in the case of compactifiable foliations we have an analog of Proposition 2.3 and its corollaries:

Proposition 5.1. *Let ω be a generic form defining compactifiable foliation and $U_i = U(\gamma_i)$ be mutually disjoint regular neighborhoods of its singular leaves. Then there exists a neighborhood $\mathcal{O}(\omega) \subset F(\Omega)$ such that any $\omega' \in \mathcal{O}(\omega)$ is also generic, with the same number of singular leaves $\gamma_i' \subset U_i$ and with a compactifiable foliation.*

P r o o f. By Theorem 4.1, there exists a neighborhood $\mathcal{O}_c(\omega) \subset F(\Omega)$ such that all forms $\omega' \in \mathcal{O}_c(\omega)$ also have compactifiable foliations.

Consider a singular leaf γ_i of ω; it is compact and contains a unique singularity $p_i \in \operatorname{Sing}\omega$; see Figure 4. Since U_i is homotopically equivalent to γ_i and $\omega|_{\gamma_i} = 0$, we have $\omega = dF_i$ in U_i. Without loss of generality assume that $\gamma_i = F_i^{-1}(0)$ and $U_i = F_i^{-1}(-\varepsilon, \varepsilon)$ for a small $\varepsilon > 0$. Now consider a neighborhood $\mathcal{O}(p_i) \subset F_i^{-1}(-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon) \subset U_i$. By Proposition 2.3, there exists a neighborhood $\mathcal{O}(F_i) \subset C^\infty(U_i, \mathbb{R})$ such that any function $F_i' \in \mathcal{O}(F_i)$ has a unique singularity with $p_i' \in \mathcal{O}(p_i)$.

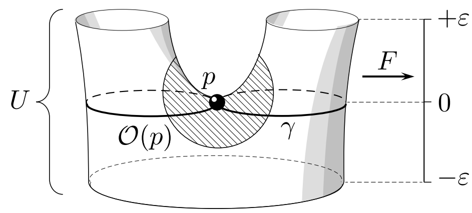

Figure 4. Left: a singularity $p \in \operatorname{Sing}\omega$ and its small neighborhood $\mathcal{O}(p)$ in a larger regular neighborhood U of the singular leaf $\gamma \ni p$. Right: slight perturbation ω' of ω, with a singularity $p' \in \operatorname{Sing}\omega'$ still in $\mathcal{O}(p)$ and its singular leaf γ' still in U.

Since $\operatorname{Sing}\omega \subset \bigcup_i U_i$, there exists $\delta > 0$ such that $\|\omega(x)\|_g > \delta$ for all $x \in M \setminus \bigcup_i U_i$. Here $\|\omega(x)\|_g = \sum g^{ij}(x)\omega_i(x)\omega_j(x)$, where g is a positive Riemannian metric globally defined on M.

Let $\mathcal{O}(\omega) = \{\omega' = \omega + df\} \subset \mathcal{O}_c(\omega)$, where functions f are small enough, i.e., $|f| < \varepsilon/4$, $\|df\|_g < \delta$, and $f + F_i \in \mathcal{O}(F_i)$ in each U_i. A form $\omega' \in \mathcal{O}(\omega)$ defines a compactifiable foliation and $\operatorname{Sing}\omega' \subset \bigcup_i U_i$. Indeed, for $p' \in \operatorname{Sing}\omega'$ it holds $\omega(p') + df(p') = 0$, and so $\|\omega(p')\|_g = \|df(p')\|_g < \delta$, which implies $p' \in \bigcup_i U_i$. By the assumption, $f + F_i \in \mathcal{O}(F_i)$ in each U_i, so $\operatorname{Sing}\omega' \subset \bigcup_i \mathcal{O}(p_i)$ and $|\operatorname{Sing}\omega'| = |\operatorname{Sing}\omega|$, with each $p_i' \in \mathcal{O}(p_i)$.

Consider a singular leaf $\gamma_i' \in \overline{\mathcal{F}}_{\omega'}$ containing p_i'. Since $p_i' \in \mathcal{O}(p_i)$, we have

$$\alpha = |F_i(p_i') + f(p_i')| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

For $x \in \gamma_i'$ we have $F_i(x) + f(x) = \alpha$, so $|F_i(x)| \leqslant |\alpha| + |f(x)| < \varepsilon$, i.e., $\gamma_i' \subset U_i$.

By the construction, each neighborhood U_i contains a unique singularity of ω', so $\gamma_i' \cap \operatorname{Sing} \omega' = \{p_i'\}$. We obtain that each singular leaf γ_i' contains a unique singularity, i.e., the form ω' is generic. □

Corollary 5.1. *For any $\Omega \in H^1(M, \mathbb{R})$, the set of generic Morse forms defining compactifiable foliation is open in $F(\Omega)$.*

Note, however, that this set, unlike the set of generic forms, is not dense in $F(\Omega)$ if $\operatorname{rk} \Omega > 1$ ([9]).

Theorem 5.1. *For any $\Omega \in H^1(M, \mathbb{R})$ and any $c \geqslant 0$, the set of generic forms with compactifiable foliation and exactly c homologically independent compact leaves is open in $F(\Omega)$.*

P r o o f. Let ω be a generic form defining compactifiable foliation $\mathcal{F}_\omega$, and $[\omega] = \Omega$. Consider sufficiently small mutually disjoint regular neighborhoods $U_i = U(\gamma_i)$ of its singular leaves. Without loss of generality assume that connected components of ∂U_i are leaves of $\mathcal{F}_\omega$, so they generate the group H_ω, i.e. $H_\omega = \langle [\partial_j U_i] \rangle$.

Consider the inclusions

$$\partial \overline{U}_i \xrightarrow{f_i} \overline{U}_i \xrightarrow{g_i} M.$$

By Lemma 2.2, we have $H_{n-1}(\overline{U}_i) = f_{i*} H_{n-1}(\partial \overline{U}_i)$; thus $H_\omega = \langle g_{i*} H_{n-1}(\overline{U}_i) \rangle$.

By Proposition 5.1, there exists a neighborhood $\mathcal{O}(\omega) \subset F(\Omega)$ such that all forms in this neighborhood are generic with compactifiable foliation; moreover, for $\omega' \in \mathcal{O}(\omega)$ its singular leaves γ_i' lie in U_i. Denote by V_i closed regular neighborhoods of γ_i' such that $V_i \subset U_i$.

Since $\mathcal{F}_{\omega'}$ is compactifiable, connected components of ∂V_i generate $H_{\omega'}$, i.e., $H_{\omega'} = \langle [\partial_j V_i] \rangle$, and by Lemma 2.2, $H_{\omega'} = \langle g'_{i*} H_{n-1}(V_i) \rangle$, where $g_i' \colon V_i \hookrightarrow M$. Since $V_i \subset U_i$, we have $\langle g'_{i*} H_{n-1}(V_i) \rangle \subseteq \langle g_{i*} H_{n-1}(\overline{U}_i) \rangle$; thus $c(\omega') \leqslant c(\omega)$. On the other hand, Theorem 3.1 (ii) gives $c(\omega') \geqslant c(\omega)$; we obtain $c(\omega') = c(\omega)$. □

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