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REMARKS ON LOCAL LIE ALGEBRAS OF PAIRS OF FUNCTIONS

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Abstract. Starting by the famous paper by Kirillov, local Lie algebras of functions over smooth manifolds were studied very intensively by mathematicians and physicists. In the present paper we study local Lie algebras of pairs of functions which generate infinitesimal symmetries of almost-cosymplectic-contact structures of odd dimensional manifolds.

Keywords: almost-cosymplectic-contact structure; almost-coPoisson-Jacobi structure; infinitesimal symmetry; local Lie algebra

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1. INTRODUCTION

The concept of a *local Lie algebra* over a smooth manifold was defined by Shiga in [14], as follows. Let $p\colon \mathbf{E} \to \mathbf{M}$ be a smooth vector bundle over a smooth manifold $\mathbf{M}$. Let us denote by $\Gamma(\mathbf{E})$ the sheaf of (local) sections of $\mathbf{E}$. A local Lie algebra over $\mathbf{M}$ is a Lie algebra in $\Gamma(\mathbf{E})$ given by the bracket (bilinear, antisymmetric and satisfying the Jacobi identity) $[s_1, s_2]$, $s_1, s_2 \in \Gamma(\mathbf{E})$, satisfying a continuity condition and $\operatorname{supp}([s_1, s_2]) \subset \operatorname{supp}(s_1) \cap \operatorname{supp}(s_2)$.

It is very well known (see [10]) that (1-dimensional) local Lie algebras of functions on a $(2n+1)$-dimensional manifold $\mathbf{M}$ are in one-to-one correspondence with *Jacobi structures* on $\mathbf{M}$ given by a skew symmetric 2-vector field Λ and a vector field E such that

$$[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = -2E \wedge \Lambda,$$

where $[,]$ is the Schouten-Nijenhuis bracket (see, for example, [16]) of skew symmetric multi-vector fields.

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Then the *Jacobi bracket*

$$[f, h] = \{f, h\} - fE.h + hE.f,$$

where $\{f, h\} = \Lambda(df, dh)$ is the *Poisson bracket*, defines on the sheaf of functions $C^\infty(\mathbf{M})$ the structure of a local (Jacobi) Lie algebra.

Let us assume the subsheaf $C_E^\infty(\mathbf{M})$ of functions constant on integral curves of the vector field E, i.e. such that $E.f = 0$ (conserved functions in the terminology of [9]). Then the restriction of the above Jacobi bracket defines the Lie subalgebra $(C_E^\infty(\mathbf{M}); \{,\}) \subset (C^\infty(\mathbf{M}); [,])$ of conserved functions. Indeed, if $f, h \in C_E^\infty(\mathbf{M})$, then

$$E.\{f, h\} = \{E.f, h\} + \{f, E.h\} = 0.$$

Moreover, the *Hamilton-Jacobi lift*

$$X_f = df^\sharp - fE$$

of a function $f \in C_E^\infty(\mathbf{M})$ is an infinitesimal symmetry of (E, Λ), i.e. $L_{X_f}E = [X_f, E] = 0$ and $L_{X_f}\Lambda = [X_f, \Lambda] = 0$. In what follows we shall use the notation $\Lambda^\sharp(\alpha) := \alpha^\sharp = i_\alpha\Lambda$ for any 1-form α.

If the pair (E, Λ) is *regular* (*transitive* in the terminology of [10]), i.e. $E \wedge \Lambda^n \not\equiv 0$, then there exists the unique 1-form ω such that $i_E\omega = 1$ and $i_\omega\Lambda = 0$. Moreover, ω is a *contact form*, i.e. $\omega \wedge d\omega^n \not\equiv 0$ and E is the Reeb vector field of the contact structure $(\omega, \Omega = d\omega)$, i.e. $i_E\Omega = 0$. The pairs (ω, Ω) and (E, Λ) are said to be *mutually dual*. It is easy to see that the Hamilton-Jacobi lift of a conserved function is an infinitesimal symmetry of the contact pair (ω, Ω), i.e. $L_{X_f}\omega = 0$ and $L_{X_f}\Omega = 0$. The Hamilton-Jacobi lift of conserved functions is a Lie algebra homomorphism from the Lie algebra $(C_E^\infty(\mathbf{M}), \{,\})$ to the Lie algebra $(\mathcal{L}(\omega, \Omega), [,]) \subset (\mathcal{X}(\mathbf{M}), [,])$ of infinitesimal symmetries of the contact structure (ω, Ω) (respective to the Lie algebra $(\mathcal{L}(E, \Lambda), [,])$ of infinitesimal symmetries of the Jacobi structure (E, Λ)). Here $(\mathcal{X}(\mathbf{M}), [,])$ is the Lie algebra of vector fields.

In [8] dual contact and Jacobi structures were generalized in the following sense. An *almost-cosymplectic-contact (regular) structure (pair)* is given by a pair (ω, Ω) such that

$$d\Omega = 0, \quad \omega \wedge \Omega^n \not\equiv 0.$$

According to [11] there exists a unique dual *almost-coPoisson-Jacobi structure (pair)* given by a pair (E, Λ) such that

$$(\Omega^\flat{}_{|\mathrm{Im}(\Lambda^\sharp)})^{-1} = \Lambda^\sharp{}_{|\mathrm{Im}(\Omega^\flat)}, \quad i_E\omega = 1, \quad i_E\Omega = 0, \quad i_\omega\Lambda = 0,$$

where $\Omega^\flat \colon T\mathbf{M} \to T^*\mathbf{M}$ is given by $\Omega^\flat(X) := X^\flat = i_X\Omega$. Then (see [8])

$$[E, \Lambda] = -E \wedge \Lambda^\sharp(L_E\omega), \qquad [\Lambda, \Lambda] = 2E \wedge (\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega).$$

Remark 1.1. The almost-cosymplectic-contact pair and the dual almost-coPoisson-Jacobi pair generalize not only the contact and the dual Jacobi structures but also cosymplectic structures. Indeed, if $d\omega = 0$, we obtain a *cosymplectic pair* (see, for example, [1]). The corresponding dual pair is *coPoisson pair* (see [8]) given by the pair (E, Λ) such that $[E, \Lambda] = 0$, $[\Lambda, \Lambda] = 0$.

Remark 1.2. For cosymplectic and contact structures we have distinguished forms ω, Ω and vector fields E, Λ. But for an almost-cosymplectic-contact structure we have distinguished forms ω, $L_E\omega$, Ω, $d\omega$ and distinguished vector fields E, $(L_E\omega)^\sharp$, Λ, $(\Lambda^\sharp \otimes \Lambda^\sharp)(d\omega)$.

In what follows we assume an odd dimensional manifold $\mathbf{M}$ with a regular almost-cosymplectic-contact structure (ω, Ω). We assume the dual (regular) almost-coPoisson-Jacobi structure (E, Λ). Then we have $\mathrm{Ker}(\omega) = \mathrm{Im}(\Lambda^\sharp)$ and $\mathrm{Ker}(E) = \mathrm{Im}(\Omega^\flat)$ and we have the splitting

$$T\mathbf{M} = \mathrm{Im}(\Lambda^\sharp) \oplus \langle E\rangle, \quad T^*\mathbf{M} = \mathrm{Im}(\Omega^\flat) \oplus \langle\omega\rangle,$$

i.e. any vector field X and any 1-form β can be decomposed as

$$X = X_{(\alpha,h)} = \alpha^\sharp + hE, \quad \beta = \beta_{(Y,f)} = Y^\flat + f\omega, \tag{1.1}$$

where $h, f \in C^\infty(\mathbf{M})$, α is a 1-form and Y is a vector field. Moreover, $h = \omega(X_{(\alpha,h)})$ and $f = \beta_{(Y,f)}(E)$. Let us note that the splitting (1.1) is not defined uniquely; indeed $X_{(\alpha_1,h_1)} = X_{(\alpha_2,h_2)}$ if and only if $\alpha_1^\sharp = \alpha_2^\sharp$ and $h_1 = h_2$, i.e. $\alpha_1^\sharp - \alpha_2^\sharp = 0$, which means that $\alpha_1 - \alpha_2 \in \langle\omega\rangle$. Similarly, $\beta_{(Y_1,f_1)} = \beta_{(Y_2,f_2)}$ if and only if $Y_1 - Y_2 \in \langle E\rangle$ and $f_1 = f_2$.

The projections $p_2 \colon T\mathbf{M} \to \langle E\rangle$ and $p_1 \colon T\mathbf{M} \to \mathrm{Im}(\Lambda^\sharp) = \mathrm{Ker}(\omega)$ are given by $X \mapsto \omega(X)E$ and $X \mapsto X - \omega(X)E$. Equivalently, the projections $q_2 \colon T^*\mathbf{M} \to \langle\omega\rangle$ and $q_1 \colon T^*\mathbf{M} \to \mathrm{Im}(\Omega^\flat) = \mathrm{Ker}(E)$ are given by $\beta \mapsto \beta(E)\omega$ and $\beta \mapsto \beta - \beta(E)\omega$. Moreover, $\Lambda^\sharp \circ \Omega^\flat = p_1$ and $\Omega^\flat \circ \Lambda^\sharp = q_1$.

Remark 1.3. Let us consider the pair (ω, F), where $F = \Omega + d\omega$ is a closed 2-form. Then the pair (ω, F), is not generally an almost-cosymplectic-contact pair because it may not be regular. We shall use the form F later in Theorem 3.1.

In [5] we have studied infinitesimal symmetries of tensor fields ω, Ω, E, Λ generating the almost-cosymplectic-contact and the dual almost-coPoisson-Jacobi structures. Such symmetries are vector fields of the type $X_{(\alpha,h)} = \alpha^\sharp + hE$, where α and h meet certain conditions and the fact that they generate infinitesimal symmetries of a tensor field defines a Lie algebra structure on some subsheaf of $\Omega^1(\mathbf{M}) \times C^\infty(\mathbf{M})$ of generators (α, h) of infinitesimal symmetries.

In this paper we shall study the situation where the 1-form α is locally exact, i.e. $\alpha = df$ for $f \in C^\infty(\mathbf{M})$. This leads to local Lie algebras of pairs of functions generating infinitesimal symmetries of some tensor fields. Such Lie algebras are 2-dimensional local Lie algebras in the sense of [10], [14].

Generally, we can define local Lie algebra structure in $C^\infty(\mathbf{M})\times C^\infty(\mathbf{M})$ by a (double) bracket $[\![(f_1,h_1);(f_2,h_2)]\!]$ in $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ satisfying the following conditions:

(1) it defines a Lie algebra structure in $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ over $\mathbb{R}$,
(2) $[\![(f_1,h_1);(f_2,h_2)]\!]$ is continuous in f_i and h_i, $i = 1, 2$,
(3) $\operatorname{supp}[\![(f_1,h_1);(f_2,h_2)]\!] \subset \operatorname{supp}(f_1,h_1) \cap \operatorname{supp}(f_2,h_2)$ for each (f_1,h_1) and (f_2,h_2), where $\operatorname{supp}(f_i,h_i) = \operatorname{supp}(f_i) \cap \operatorname{supp}(h_i)$.

As a simple example we can consider two Jacobi pairs (E_i, Λ_i), $i = 1, 2$, and the corresponding Jacobi brackets $[,]_i$. Then the bracket

$$[\![(f_1,h_1);(f_2,h_2)]\!] = ([f_1,f_2]_1;[h_1,h_2]_2)$$

defines the Lie algebra structure in $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$. As far as we know, the classification of local Lie algebras of pairs of functions is not known. In the paper we shall describe several subsheafs of $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ with a Lie algebra structure given by the fact that pairs of functions generate infinitesimal symmetries of basic tensor fields $\omega, \Omega, E, \Lambda$ given by the almost-cosymplectic-contact and the dual almost-coPoisson-Jacobi structures.

All manifolds and mappings are assumed to be smooth.

2. Local Lie algebras of generators of infinitesimal symmetries

In this section we shall study local Lie algebras of pairs of functions which generate infinitesimal symmetries of basic tensor fields.

For two functions $f, h \in C^\infty(\mathbf{M})$ we define their *pre-Hamiltonian lift* to vector fields on $\mathbf{M}$ by

$$X_{(f,h)} = df^\sharp + hE. \tag{2.1}$$

Lemma 2.1. *Let $(f_i, h_i) \in C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$, $i = 1, 2$, be two pairs of functions on* $\mathbf{M}$. *Then*

$$
\begin{aligned}
(2.2)\qquad [X_{(f_1,h_1)}, X_{(f_2,h_2)}] = \big(&d\{f_1, f_2\} + (E.f_1)(L_{df_2^\sharp} + h_2 L_E)\omega \\
&- (E.f_2)(L_{df_1^\sharp} + h_1 L_E)\omega - h_2 d(E.f_1) + h_1 d(E.f_2)\big)^\sharp \\
&+ \big(\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp) \\
&+ h_1(E.h_2 + \Lambda(L_E\omega, df_2)) - h_2(E.h_1 + \Lambda(L_E\omega, df_1))\big)E.
\end{aligned}
$$

P r o o f. We have (see [8])

$$
(2.3)\qquad [E, df^\sharp] = \big(d(E.f) - (E.f)(L_E\omega)\big)^\sharp + \Lambda(L_E\omega, df)E,
$$

$$
\begin{aligned}
(2.4)\qquad [df^\sharp, dh^\sharp] = \big(&d\Lambda(df, dh) + (E.f)(i_{dh^\sharp} d\omega) \\
&- (E.h)(i_{df^\sharp} d\omega)\big)^\sharp - d\omega(df^\sharp, dh^\sharp)E,
\end{aligned}
$$

which implies (2.2). □

It is easy to see that the vector field (2.2) is not generally the pre-Hamiltonian lift of a pair of functions. It is so in the case when the projection $p_1 \colon T\mathbf{M} \to \mathrm{Ker}(\omega)$ of (2.2) is the $\Lambda^\sharp$-lift of the differential of a function. We shall describe several examples of subsheafs of $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ such that the Lie bracket of two pre-Hamiltonian lifts of pairs of functions from the subsheaf is the pre-Hamiltonian lift of a pair from the subsheaf. All such subsheafs are given by local generators of infinitesimal symmetries of basic tensor fields.

2.1. Infinitesimal symmetries of ω and Ω generated by pairs of functions.

Theorem 2.2. *The pre-Hamiltonian lift* (2.1) *is an infinitesimal symmetry of* ω *if and only if*

$$
(2.5)\qquad i_{df^\sharp} d\omega + h i_E d\omega + dh = 0.
$$

P r o o f. From $i_E\omega = 1$ and $i_{df^\sharp}\omega = 0$ we get

$$
L_{X_{(f,h)}}\omega = i_{df^\sharp} d\omega + h i_E d\omega + dh,
$$

which proves Theorem 2.2. □

Let us denote by $\mathtt{LGen}(\omega) \subset C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ the subsheaf of pairs of functions (f, h) on $\mathbf{M}$ which generate (locally) infinitesimal symmetries of ω, i.e. satisfy condition (2.5).

Theorem 2.3. *The Lie bracket of the pre-Hamiltonian lifts of two pairs* $(f_i, h_i) \in \mathtt{LGen}(\omega)$, $i = 1, 2$, *is the pre-Hamiltonian lift of a pair of functions from* $\mathtt{LGen}(\omega)$.

Proof. If (2.5) is satisfied, then $dh_i = -(i_{df_i^\sharp} + h_i i_E) d\omega = -(L_{df_i^\sharp} + h_i L_E)\omega$ and $E.h_i + \Lambda(L_E\omega, df_i) = 0$, which follows by evaluating (2.5) on E. Then we can rewrite (2.2) as

$$\begin{aligned}(2.6)\qquad [X_{(f_1,h_1)}, X_{(f_2,h_2)}] = {} & \big(d(\{f_1, f_2\} - h_2(E.f_1) + h_1(E.f_2))\big)^\sharp \\ & + \big(\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big)E,\end{aligned}$$

which is the pre-Hamiltonian lift of the pair

$$(2.7)\qquad \big(\{f_1, f_2\} - h_2(E.f_1) + h_1(E.f_2); \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big).$$

The pair (2.7) is in $\mathtt{LGen}(\omega)$, which follows from the fact that, according to Theorem 2.2, the pre-Hamiltonian lifts (2.1) of pairs from $\mathtt{LGen}(\omega)$ are infinitesimal symmetries of ω. Then from $L_{[X,Y]} = L_X L_Y - L_Y L_X$, the Lie bracket (2.6) of two pre-Hamiltonian lifts of pairs from $\mathtt{LGen}(\omega)$ is an infinitesimal symmetry of ω and, by Theorem 2.2, the pair (2.7) has to satisfy condition (2.5), i.e. it is in $\mathtt{LGen}(\omega)$. □

As a consequence we obtain the Lie bracket

$$\begin{aligned}(2.8)\qquad [\![(f_1, h_1); (f_2, h_2)]\!] = {} & \big(\{f_1, f_2\} - h_2(E.f_1) + h_1(E.f_2); \\ & \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big),\end{aligned}$$

which defines the local Lie algebra structure on $\mathtt{LGen}(\omega)$. Moreover, the pre-Hamiltonian lift (2.1) is the Lie algebra homomorphism from the local Lie algebra $(\mathtt{LGen}(\omega); [\![,]\!])$ to the Lie algebra $(\mathcal{L}(\omega); [,]) \subset (\mathcal{X}(\mathbf{M}); [,])$ of infinitesimal symmetries of ω.

Now, we shall define a Lie algebra structure on the subsheaf $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$ of pairs (f, h) of functions, where f is conserved.

Theorem 2.4. *The Lie bracket of the pre-Hamiltonian lifts of two pairs* $(f_i, h_i) \in C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$, $i = 1, 2$, *is the pre-Hamiltonian lift of a pair of functions from* $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$.

P r o o f. For $E.f_i = 0$ the vector field (2.2) is expressed as

$$[X_{(f_1,h_1)}, X_{(f_2,h_2)}] = \big(d\{f_1,f_2\}\big)^\sharp + \big(\{f_1,h_2\} - \{f_2,h_1\} - d\omega(df_1^\sharp, df_2^\sharp) \\ + h_1(E.h_2 + \Lambda(L_E\omega, df_2)) - h_2(E.h_1 + \Lambda(L_E\omega, df_1))\big)E.$$

This vector field is the pre-Hamiltonian lift (2.1) of the pair

$$\big(\{f_1,f_2\}; \{f_1,h_2\} - \{f_2,h_1\} - d\omega(df_1^\sharp, df_2^\sharp) \\ + h_1(E.h_2 + \Lambda(L_E\omega, df_2)) - h_2(E.h_1 + \Lambda(L_E\omega, df_1))\big).$$

Moreover, the above pair is in $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$. Indeed,

$$E.\{f_1,f_2\} = \{E.f_1, f_2\} + \{f_1, E.f_2\} + i_{[E,\Lambda]}(df_1 \wedge df_2) \\ = -i_{E\wedge(L_E\omega)^\sharp}(df_1 \wedge df_2) = 0,$$

which proves Theorem 2.4. □

As a consequence, by observing $\Lambda(L_E\omega, df) = (L_E\omega)^\sharp.f$, we obtained the Lie bracket

$$\llbracket (f_1,h_1);(f_2,h_2)\rrbracket = \big(\{f_1,f_2\}; \{f_1,h_2\} - \{f_2,h_1\} - d\omega(df_1^\sharp, df_2^\sharp) \\ + h_1(E.h_2 + (L_E\omega)^\sharp.f_2) - h_2(E.h_1 + (L_E\omega)^\sharp.f_1)\big) \tag{2.9}$$

of pairs of functions from $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$, which defines the local Lie algebra structure on $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$.

In [2], [5] it was proved that all infinitesimal symmetries of Ω are vector fields $X_{(\alpha,h)} = \alpha^\sharp + hE$, where α is a closed 1-form such that $\alpha(E) = 0$. Hence, locally $\alpha = df$ for a function $f \in C^\infty_E(\mathbf{M})$ and any infinitesimal symmetry of Ω is locally the pre-Hamiltonian lift of a pair of functions from $C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M})$. So the Lie algebra of local generators of infinitesimal symmetries of Ω is $(\texttt{LGen}(\Omega), \llbracket,\rrbracket) \equiv (C^\infty_E(\mathbf{M}) \times C^\infty(\mathbf{M}), \llbracket,\rrbracket)$ with the Lie bracket (2.9). The pre-Hamiltonian lift (2.1) is the Lie algebra homomorphism from the local Lie algebra $(\texttt{LGen}(\Omega); \llbracket,\rrbracket)$ to the Lie algebra $(\mathcal{L}(\Omega); [,]) \subset (\mathcal{X}(\mathbf{M}); [,])$ of infinitesimal symmetries of Ω.

Theorem 2.5. *A vector field $X_{(f,h)}$ is an infinitesimal symmetry of the almost-cosymplectic-contact structure (ω,Ω) if and only if $f \in C^\infty_E(\mathbf{M})$ and condition* (2.5) *is satisfied.*

P r o o f. It follows from Theorems 2.2 and the fact that $X_{(f,h)}$ is an infinitesimal symmetry of Ω if and only if $f \in C^\infty_E(\mathbf{M})$. □

Lemma 2.6. *A vector field $X_{(f,h)}$ is an infinitesimal symmetry of (ω,Ω) if and only if the following conditions are satisfied:*

(1) $E.f = i_E df = 0$,

(2) $i_E dh + i_E i_{df^\sharp} d\omega = E.h + (L_E\omega)^\sharp.f = 0$,

(3) $d\omega(df^\sharp,\beta^\sharp) + h d\omega(E,\beta^\sharp) + dh(\beta^\sharp) = 0$ *for any 1-form β, especially, if we put $\beta = dg$ for a $g \in C^\infty(\mathbf{M})$, we get* $\{g,h\} = d\omega(dg^\sharp, df^\sharp) + h d\omega(dg^\sharp, E)$.

Proof. It is a consequence of Theorem 2.5 and Theorem 2.2, where we have evaluated the 1-form on the left hand side of (2.5) on E (which gives condition (2)) and on $\beta^\sharp$ for any 1-form β (which gives condition (3)). $\square$

We denote the sheaf of pairs of functions which locally generate infinitesimal symmetries of the almost-cosymplectic-contact structure (ω,Ω) as $\mathtt{LGen}(\omega,\Omega) = \mathtt{LGen}(\omega) \cap \mathtt{LGen}(\Omega)$. Brackets (2.8) and (2.9) restricted for generators of infinitesimal symmetries of (ω,Ω) give the equivalet expressions of the bracket

$$\begin{aligned}(2.10)\quad [\![(f_1,h_1);(f_2,h_2)]\!] &= \big(\{f_1,f_2\};\{f_1,h_2\} - \{f_2,h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big)\\ &= \big(\{f_1,f_2\}; d\omega(df_1^\sharp, df_2^\sharp) + h_2 (L_E\omega)^\sharp.f_1 - h_1 (L_E\omega)^\sharp.f_2\big)\\ &= \big(\{f_1,f_2\}; d\omega(df_1^\sharp, df_2^\sharp) + h_1 E.h_2 - h_2 E.h_1\big),\end{aligned}$$

which defines the Lie algebra structure on $\mathtt{LGen}(\omega,\Omega)$.

Corollary 2.7. *An infinitesimal symmetry of the cosymplectic structure (ω,Ω) is of local type $X_{(f,h)} = df^\sharp + hE$, where $f \in C^\infty_E(\mathbf{M})$ and h is a constant.*

Then bracket (2.10) is reduced to

$$[\![(f_1,h_1);(f_2,h_2)]\!] = \big(\{f_1,f_2\},0\big).$$

I.e. we obtain the subalgebra $(\mathtt{LGen}_{cos}(\omega,\Omega);[\![,]\!]) = (C^\infty_E(\mathbf{M});\{,\}) \oplus (\mathbb{R};[,])$ of local generators of infinitesimal symmetries of the cosymplectic structure. Here $[,]$ is the trivial Lie bracket in $\mathbb{R}$.

Proof. It follows from $d\omega = 0$. Considering this from (2.5) we get $dh = 0$. $\square$

Corollary 2.8. *Any infinitesimal symmetry of the contact structure $(\omega,\Omega = d\omega)$ is of local type*

$$X_{(f,-f)} = df^\sharp - fE,$$

where $f \in C^\infty_E(\mathbf{M})$.

Then bracket (2.10) is reduced to

$$[\![(f_1,-f_1);(f_2,-f_2)]\!] = \big(\{f_1,f_2\}, -\{f_1,f_2\}\big).$$

I.e. we obtain the subalgebra $(\mathtt{LGen}_{con}(\omega); [\![,]\!]) \subset (\mathtt{LGen}(\omega,\Omega); [\![,]\!])$ *of local generators of infinitesimal symmetries of the contact structure. Moreover,* $(\mathtt{LGen}_{con}(\omega); [\![;]\!]) \equiv (C^\infty_E(\mathbf{M}), \{,\})$.

Proof. It follows from $d\omega = \Omega$, $i_E\Omega = 0$ and $i_{df^\sharp}\Omega = df$. Considering this from (2.5) we get $df = -dh$. □

2.2. Infinitesimal symmetries of E and Λ generated by pairs of functions.

Theorem 2.9. *The pre-Hamiltonian lift* (2.1) *is an infinitesimal symmetry of the Reeb vector field* E *if and only if*

$$\big(d(E.f) - (E.f)L_E\omega\big)^\sharp = 0, \tag{2.11}$$

$$(E.h) + (L_E\omega)^\sharp.f = 0. \tag{2.12}$$

Proof. Let us assume that the pre-Hamiltonian lift of a pair (f,h) is an infinitesimal symmetry of the Reeb vector field. Then from (2.3)

$$0 = [X_{(f,h)}, E] = -\big(d(E.f) - (E.f)L_E\omega\big)^\sharp - ((E.h) + (L_E\omega)^\sharp.f)E,$$

which proves Theorem 2.9. □

If we have two pairs (f_i, h_i) generating infinitesimal symmetries of E, then the Lie bracket of the corresponding pre-Hamiltonian lifts is

$$\begin{aligned}[X_{(f_1,h_1)}, X_{(f_2,h_2)}] = {} & \big(d\{f_1,f_2\} + (E.f_1)L_{df_2^\sharp}\omega - (E.f_2)L_{df_1^\sharp}\omega\big)^\sharp \\ & + \big(\{f_1,h_2\} - \{f_2,h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big)E.\end{aligned} \tag{2.13}$$

The above vector field (2.13) is not generally the pre-Hamiltonian lift of a pair of functions. So, the sheaf of local generators of infinitesimal symmetries of E is not a Lie algebra.

Theorem 2.10. *The pre-Hamiltonian lift* (2.1) *is an infinitesimal symmetry of* Λ, *i.e.* $L_{X_{(f,h)}}\Lambda = [X_{X_{(f,h)}}, \Lambda] = 0$ *if and only if the following condition is satisfied:*

$$[df^\sharp, \Lambda] - E \wedge (dh + hL_E\omega)^\sharp = 0. \tag{2.14}$$

Proof. We have

$$L_{X_{(f,h)}}\Lambda = [df^\sharp, \Lambda] + [hE, \Lambda].$$

Theorem 2.10 follows from

$$[hE, \Lambda] = h[E, \Lambda] - E \wedge dh^\sharp = -E \wedge (dh + hL_E\omega)^\sharp.$$

$\square$

Lemma 2.11. *A vector field $X_{(f,h)}$ is an infinitesimal symmetry of Λ if and only if conditions*

$$E.f = 0, \tag{2.15}$$

$$d\omega(df^\sharp, \beta^\sharp) + hd\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0, \tag{2.16}$$

are satisfied for any 1-form β.

Proof. It is sufficient to evaluate the 2-vector field on the left hand side of (2.14) on ω, β and β, γ, where β, γ are closed 1-forms. We get

$$i_{[df^\sharp,\Lambda]-E\wedge(dh+hL_E\omega)^\sharp}(\omega\wedge\beta) = -\Lambda(i_{df^\sharp}d\omega + hL_E\omega + dh, \beta),$$

which vanishes if and only if (2.16) is satisfied.

On the other hand,

$$\begin{aligned} & i_{[df^\sharp,\Lambda]-E\wedge(dh+hL_E\omega)^\sharp}(\beta\wedge\gamma) \\ &\qquad = \Lambda(df, d\Lambda(\beta,\gamma)) + \Lambda(\beta, d\Lambda(\gamma, df)) + \Lambda(\gamma, d\Lambda(df,\beta)) \\ &\qquad\quad - \beta(E)\Lambda(hL_E\omega + dh, \gamma) + \gamma(E)\Lambda(hL_E\omega + dh, \beta), \end{aligned}$$

which, using (2.16), can be rewritten as

$$\begin{aligned} i_{[df^\sharp,\Lambda]-E\wedge(dh+hL_E\omega)^\sharp}(\beta\wedge\gamma) = & -\frac{1}{2}i_{[\Lambda,\Lambda]}(df\wedge\beta\wedge\gamma) + \beta(E)\Lambda(i_{df^\sharp}d\omega, \gamma) \\ & - \gamma(E)\Lambda(i_{df^\sharp}d\omega, \beta) \\ = & - i_{E\wedge(\Lambda^\sharp\otimes\Lambda^\sharp)(d\omega)}(df\wedge\beta\wedge\gamma) \\ & + \beta(E)\Lambda(i_{df^\sharp}d\omega,\gamma) - \gamma(E)\Lambda(i_{df^\sharp}d\omega,\beta) \\ = & - df(E)d\omega(\beta^\sharp,\gamma^\sharp) = -(E.f)d\omega(\beta^\sharp,\gamma^\sharp), \end{aligned}$$

which vanishes if and only if (2.15) is satisfied.

On the other hand, if (2.15) and (2.16) are satisfied, then the 2-vector field $L_{X_{(f,h)}}\Lambda$ is the zero 2-vector field. $\square$

So, local generators of infinitesimal symmetries of Λ are from $\mathtt{LGen}(\Omega)$ and generate also infinitesimal symmetries of Ω, i.e. $\mathtt{LGen}(\Lambda) \subset \mathtt{LGen}(\Omega)$ and the Lie algebra of local generators of infinitesimal symmetries of Λ is the subalgebra $(\mathtt{LGen}(\Lambda); [\![,]\!]) \subset (\mathtt{LGen}(\Omega); [\![,]\!])$ with the bracket

$$[\![(f_1, h_1); (f_2, h_2)]\!] = \big(\{f_1, f_2\}; d\omega(df_1^\sharp, df_2^\sharp) + h_1 E.h_2 - h_2 E.h_1\big),$$

which we obtain from bracket (2.9) restricted for functions satisfying (2.16).

Corollary 2.12. *The Lie algebra of local generators of infinitesimal symmetries of the almost-coPoisson-Jacobi pair* (E, Λ) *coincides with the local Lie algebra of generators of infinitesimal symmetries of* (ω, Ω).

P r o o f. From Theorem 2.9 and Lemma 2.11 the pre-Hamiltonian lift (2.1) is an infinitesimal symmetry of the pair (E, Λ) if and only if conditions (1), (2) and (3) of Lemma 2.6 are satisfied, i.e. if and only if it is an infinitesimal symmetry of the pair (ω, Ω). $\square$

Remark 2.1. Let us consider subsheaves of $C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ given by conditions (1), (2) and (3) of Lemma 2.6. The subsheaf $\mathtt{LGen}(\Omega)$ is given by condition (1), the subsheaf $\mathtt{LGen}(\omega)$ is given by conditions (2) and (3) and the subsheaf $\mathtt{LGen}(\Lambda)$ is given by conditions (1) and (3). If we assume the subsheaf given by conditions (1) and (2), then from Theorem 2.9 it is the subsheaf $\mathtt{LGen}(E, \Omega)$ of local generators of infinitesimal symmetries of the Reeb vector field E and Ω. The corresponding bracket will be

$$[\![(f_1, h_1); (f_2, h_2)]\!] = \big(\{f_1, f_2\}; \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big).$$

So we obtain the following local Lie algebras of generators of infinitesimal symmetries:

$(\mathtt{LGen}(\Omega), [\![,]\!])$	(1), $E.f = 0$
$(\mathtt{LGen}(\omega), [\![,]\!])$	(2), $E.h + (L_E\omega)^\sharp.f = 0$
	(3), $d\omega(df^\sharp, \beta^\sharp) + h d\omega(E, \beta^\sharp) + dh(\beta^\sharp) = 0$
$(\mathtt{LGen}(\Lambda), [\![,]\!])$	(1) and (3)
$(\mathtt{LGen}(E, \Omega), [\![,]\!])$	(1) and (2)
$(\mathtt{LGen}(\omega, \Omega), [\![,]\!]) \equiv (\mathtt{LGen}(E, \Lambda), [\![,]\!])$	(1), (2) and (3)

Remark 2.2. In the Lie algebra $(\mathtt{LGen}(\Omega); [\![,]\!])$ we have the Abelian subalgebra formed by pairs of constant functions $\mathcal{K}(\mathbf{M}) = \mathbb{R} \times \mathbb{R} \subset \mathtt{LGen}(\Omega)$. The centralizer of the Lie subalgebra $\mathcal{K}(\mathbf{M})$ in $\mathtt{LGen}(\Omega)$ is the Lie subalgebra $(\mathtt{LGen}(E, \Omega); [\![,]\!]) \subset (\mathtt{LGen}(\Omega); [\![,]\!])$ of generators of infinitesimal symmetries of E and Ω.

Indeed, let $(c,k) \in \mathcal{K}(\mathbf{M})$ and $(f,h) \in \texttt{LGen}(\Omega)$ such that

$$(0;0) = [\![(c,k);(f,h)]\!] = \big(0; -k(E.h + (L_E\omega)^\sharp.f)\big).$$

Then from Theorem 2.9 it follows that the pair (f,h) generates an infinitesimal symmetry of E.

2.3. Multiplicative algebra $(\texttt{LGen}(\Omega), \cdot)$**.**

In Section 2.1 we have defined the local Lie algebra structure on $\texttt{LGen}(\Omega)$. We define the multiplication in $\texttt{LGen}(\Omega)$ by

$$(f_1,h_1)(f_2,h_2) = (f_1f_2, f_1h_2 + f_2h_1),$$

which defines on $\texttt{LGen}(\Omega)$ the structure of an associative commutative algebra with the unit $(1,0)$. Indeed, it is easy to see that

$$\begin{gathered}(f_1,h_1)(f_2,h_2) = (f_2,h_2)(f_1,h_1),\\ \big((f_1,h_1)(f_2,h_2)\big)(f_3,h_3) = (f_1,h_1)\big((f_2,h_2)(f_3,h_3)\big),\\ (1,0)(f,h) = (f,h).\end{gathered}$$

Lemma 2.13. *We have*

$$X_{(f_1f_2, f_1h_2+f_2h_1)} = f_1X_{(f_2,h_2)} + f_2X_{(f_1,h_1)}.$$

P r o o f. We have

$$\begin{aligned}X_{(f_1f_2,f_1h_2+f_2h_1)} &= d(f_1f_2)^\sharp + (f_1h_2 + f_2h_1)E\\ &= f_1(df_2^\sharp + h_2E) + f_2(df_1^\sharp + h_1E)\\ &= f_1X_{(f_2,h_2)} + f_2X_{(f_1,h_1)}.\end{aligned}$$

□

From (2.9) it is easy to see that the bidifferential operator D on $\texttt{LGen}(\Omega)$ given by

$$D_{(f_1,h_1)}(f_2,h_2) = [\![(f_1,h_1);(f_2,h_2)]\!]$$

is of order 1.

From the Jacobi identity we get

$$D_{[\![(f_1,h_1);(f_2,h_2)]\!]}(f_3,h_3) = \big(D_{(f_1,h_1)}D_{(f_2,h_2)} - D_{(f_2,h_2)}D_{(f_1,h_1)}\big)(f_3,h_3)$$

and

$$D_{(f_1,h_1)}[\![(f_2,h_2);(f_3,h_3)]\!] = [\![D_{(f_1,h_1)}(f_2,h_2);(f_3,h_3)]\!] + [\![(f_2,h_2);D_{(f_1,h_1)}(f_3,h_3)]\!],$$

i.e. $D_{(f,h)}$ is a derivation on $(\texttt{LGen}(\Omega);[\![,]\!])$.

Theorem 2.14. *The 1st order differential operator*

$$D_{(f,h)}\colon \texttt{LGen}(\Omega) \to \texttt{LGen}(\Omega)$$

is a derivation on $(\texttt{LGen}(\Omega),\cdot)$, *i.e. for* $(f_i,h_i) \in \texttt{LGen}(\Omega)$, $i=1,2,3$, *we have*

$$D_{(f_1,h_1)}\big((f_2,h_2)(f_3,h_3)\big) = (f_2,h_2)D_{(f_1,h_1)}(f_3,h_3) + (f_3,h_3)D_{(f_1,h_1)}(f_2,h_2).$$

P r o o f. From (2.9) we can prove

$$\begin{aligned}[\![(f_1,h_1);(f_2,h_2)(f_3,h_3)]\!] &= (f_2,h_2)[\![(f_1,h_1);(f_3,h_3)]\!] \\ &\quad + (f_3,h_3)[\![(f_1,h_1);(f_2,h_2)]\!],\end{aligned}$$

which proves Theorem 2.14. □

2.4. Lie derivation of pairs of functions. We define the Lie derivation of pairs of functions $(f,h) \in C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$ given by a vector field X on $\mathbf{M}$ by $L_X(f,h) = (L_Xf, L_Xh) = (X.f, X.h)$. Generally L_X is not an operator on $\texttt{LGen}(\Omega)$.

Lemma 2.15. *Let X be a vector field on $\mathbf{M}$ such that $L_XE = [X,E] = 0$. Then for $(f,h) \in \texttt{LGen}(\Omega)$ the Lie derivation $L_X(f,h) \in \texttt{LGen}(\Omega)$.*

P r o o f. We have $L_Ef = E.f = 0$ and $[X,E] = 0$ which implies

$$0 = L_{[X,E]}f = L_XL_Ef - L_EL_Xf = -E.(L_Xf),$$

i.e. $L_Xf \in C^\infty_E(\mathbf{M})$ and $L_X(f,h) \in \texttt{LGen}(\Omega)$. □

Lemma 2.16. *If a vector field X is an infinitesimal symmetry of E, then L_X is a derivation on* $(\texttt{LGen}(\Omega),\cdot)$, *i.e.*

$$L_X\big((f_1,h_1)(f_2,h_2)\big) = (L_Xf_1, L_Xh_1)(f_2,h_2) + (f_1,h_1)(L_Xf_2, L_Xh_2).$$

P r o o f. By Lemma 2.15, $L_X\big((f_1,h_1)(f_2,h_2)\big) \in \texttt{LGen}(\Omega)$ and

$$\begin{aligned} L_X\big((f_1,h_1)(f_2,h_2)\big) &= L_X(f_1f_2, f_1h_2 + f_2h_1) \\ &= \big((L_Xf_1)f_2 + f_1(L_Xf_2); \\ &\qquad (L_Xf_1)h_2 + f_1(L_Xh_2) + (L_Xf_2)h_1 + f_2(L_Xh_1)\big) \\ &= (L_Xf_1, L_Xh_1)(f_2,h_2) + (f_1,h_1)(L_Xf_2, L_Xh_2). \end{aligned}$$

□

On the other hand, it is easy to see that if X is an infinitesimal symmetry of E, then L_X is not a derivation on the Lie algebra $(\texttt{LGen}(\Omega), [\![,]\!])$. But we have:

Theorem 2.17. *Let X be an infinitesimal symmetry of the almost-coPoisson-Jacobi structure (E,Λ). Then X is a Lie derivation on the Lie algebra $(\texttt{LGen}(\omega,\Omega); [\![,]\!])$.*

P r o o f. $L_XE = 0$ and $L_X\Lambda = 0$ imply

$$L_X\{f_1,f_2\} = \{L_Xf_1, f_2\} + \{f_1, L_Xf_2\},$$

$L_EL_X\omega = 0$, $L_EL_Xf = L_XL_Ef$ and $L_Xd\omega = 0$.

We have to prove that $(L_Xf, L_Xh) \in \texttt{LGen}(\omega,\Omega)$ for any $(f,h) \in \texttt{LGen}(\omega,\Omega)$. First, from Lemma 2.15, $L_Xf \in C^\infty_E(\mathbf{M})$. Further, we have to prove conditions (2) and (3) of Lemma 2.6 for the pair of functions (L_Xf, L_Xh). Condition (2) can be expressed as

$$0 = dh(E) + \Lambda(L_E\omega, df).$$

If we apply L_X on the above identity, we get from $L_Xdf = dL_Xf$,

$$\begin{aligned} 0 &= (L_Xdh)(E) + dh(L_XE) + (L_X\Lambda)(L_E\omega, df) + \Lambda(L_XL_E\omega, df) + \Lambda(L_E\omega, L_Xdf) \\ &= (dL_Xh)(E) + \Lambda(L_E\omega, dL_Xf), \end{aligned}$$

which is condition (2) for the pair (L_Xf, L_Xh).

Further, applying L_X on condition (3) we get

$$\begin{aligned} 0 &= (L_Xd\omega)(df^\sharp, \beta^\sharp) + d\omega(L_Xdf^\sharp, \beta^\sharp) + d\omega(df^\sharp, L_X\beta^\sharp) \\ &\quad + (L_Xh)d\omega(E,\beta^\sharp) + hd\omega(L_XE, \beta^\sharp) + hd\omega(E, L_X\beta^\sharp) \\ &\quad + (L_Xdh)(\beta^\sharp) + dh(L_X\beta^\sharp) \\ &= d\omega(d(L_Xf)^\sharp, \beta^\sharp) + (L_Xh)d\omega(E,\beta^\sharp) + (dL_Xh)(\beta^\sharp), \end{aligned}$$

which follows from $L_Xdf^\sharp = d(L_Xf)^\sharp$ (see [4], Lemma 2.15), and condition (3) for the pair (L_Xf, L_Xh) is satisfied. Hence $(L_Xf, L_Xh) \in \texttt{LGen}(\omega,\Omega)$.

Finally, we assume bracket (2.10) in the form

$$[\![(f_1,h_1);(f_2,h_2)]\!] = \big(\{f_1,f_2\}; d\omega(df_1^\sharp, df_2^\sharp) + h_1E.h_2 - h_2E.h_1\big).$$

Then

$$\begin{aligned}L_X[\![(f_1,h_1);(f_2,h_2)]\!] = \big(&\{L_Xf_1,f_2\} + \{f_1,L_Xf_2\}; d\omega(L_Xdf_1^\sharp, df_2^\sharp) + d\omega(df_1^\sharp, L_Xdf_2^\sharp)\\ &+ L_Xh_1L_Eh_2 + h_1L_XL_Eh_2 - L_Xh_2L_Eh_1 - h_2L_XL_Eh_1\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}&[\![(L_Xf_1,L_Xh_1);(f_2,h_2)]\!] + [\![(f_1,h_1);(L_Xf_2,L_Xh_2)]\!]\\ &\quad= \big(\{L_Xf_1,f_2\}; d\omega(d(L_Xf_1)^\sharp, df_2^\sharp) + L_Xh_1L_Eh_2 - h_2L_EL_Xh_1\big)\\ &\qquad+ \big(\{f_1,L_Xf_2\}; d\omega(df_1^\sharp, d(L_Xf_2)^\sharp) + h_1L_EL_Xh_2 - L_Xh_2L_Eh_1\big)\end{aligned}$$

and from $L_Xdf^\sharp = d(L_Xf)^\sharp$ we get

$$L_X[\![(f_1,h_1);(f_2,h_2)]\!] = [\![(L_Xf_1,L_Xh_1);(f_2,h_2)]\!] + [\![(f_1,h_1);(L_Xf_2,L_Xh_2)]\!],$$

i.e. L_X is a derivation on the Lie algebra $(\texttt{LGen}(\omega,\Omega); [\![,]\!])$. □

Remark 2.3. We have

$$[\![(f_1,h_1);(f_2,h_2)]\!] = \frac{1}{2}\big(L_{X_{(f_1,h_1)}}(f_2,h_2) - L_{X_{(f_2,h_2)}}(f_1,h_1)\big). \tag{2.17}$$

Indeed,

$$\begin{aligned}&L_{X_{(f_1,h_1)}}(f_2,h_2) - L_{X_{(f_2,h_2)}}(f_1,h_1)\\ &\qquad= \big(2\{f_1,f_2\}; \{f_1,h_2\} - \{f_2,h_1\} + h_1E.h_2 - h_2E.h_1\big)\end{aligned}$$

and from (2) and (3) of Lemma 2.6 we have

$$\{f_1,h_2\} - \{f_2,h_1\} = 2d\omega(df_1^\sharp, df_2^\sharp) + h_1E.h_2 - h_2E.h_1,$$

which implies (2.17).

3. Lie algebroid and infinitesimal symmetries

Let us assume a closed 2-form F on $\mathbf{M}$ and the vector bundle $\mathbf{E} = T\mathbf{M} \oplus_{\mathbf{M}} \mathbb{R} \to \mathbf{M}$. Then sections of $\mathbf{E}$ are pairs (X, f) of vector fields on $\mathbf{M}$ and functions on $\mathbf{M}$. We define a bracket of sections of $\mathbf{E}$ by (see [6])

$$[\![(X_1, f_1); (X_2, f_2)]\!]_F = \big([X_1, X_2]; X_1.f_2 - X_2.f_1 + F(X_1, X_2)\big). \tag{3.1}$$

This bracket defines an F-Lie algebroid structure (see, for instance, [12]) on $\mathbf{E}$ with the anchor given by the projection on the first component.

Indeed, bracket (3.1) is antisymmetric and from the closure of F the Jacobi identity is satisfied. Moreover, for any $h \in C^\infty(\mathbf{M})$ we get

$$\begin{aligned} [\![(X_1, f_1); h(X_2, f_2)]\!]_F &= [\![(X_1, f_1); (hX_2, hf_2)]\!]_F \\ &= h[\![(X_1, f_1); (X_2, f_2)]\!]_F + (X_1.h)(X_2, f_2) \end{aligned}$$

and the Leibniz-type formula is satisfied.

Now, let us assume the sheaf mapping from the local Lie algebra $(\mathtt{LGen}(\omega, \Omega); [\![,]\!])$ to sections of the F-Lie algebroid given by pairs of vector fields on $\mathbf{M}$ and functions on $\mathbf{M}$ given by

$$\mathbf{s}\colon\ (f, h) \mapsto (X_{(f,h)}, f - h). \tag{3.2}$$

Theorem 3.1. *For the closed* 2*-form*

$$F = \Omega + d\omega,$$

the sheaf mapping

$$\mathbf{s}\colon\ \mathtt{LGen}(\omega, \Omega) \to \mathcal{X}(\mathbf{M}) \times C^\infty(\mathbf{M})$$

given by (3.2) *is a Lie algebra morphism.*

Proof. For $(f_i, h_i) \in \mathtt{LGen}(\omega, \Omega)$, $i = 1, 2$, we have

$$\begin{aligned} [\![(f_1, h_1); (f_2, h_2)]\!] &= \big(\{f_1, f_2\}; \{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big) \\ &\mapsto \big(d\{f_1, f_2\}^\sharp + \big(\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big)E; \\ &\qquad \{f_1, f_2\} - \{f_1, h_2\} + \{f_2, h_1\} + d\omega(df_1^\sharp, df_2^\sharp)\big). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [\![\mathbf{s}(f_1, h_1); \mathbf{s}(f_2, h_2)]\!]_F = \big(&d\{f_1, f_2\}^\sharp + \big(\{f_1, h_2\} - \{f_2, h_1\} - d\omega(df_1^\sharp, df_2^\sharp)\big)E; \\ &2\{f_1, f_2\} - \{f_1, h_2\} + \{f_2, h_1\} - h_1 E.h_2 + h_2 E.h_1 \\ &+ F(df_1^\sharp + h_1 E, df_2^\sharp + h_2 E)\big). \end{aligned}$$

The first parts of the above pairs are equal. The second parts are equal if and only if

$$0 = \{f_1, f_2\} - d\omega(df_1^\sharp, df_2^\sharp) - h_1 E.h_2 + h_2 E.h_1 + F(df_1^\sharp + h_1 E, df_2^\sharp + h_2 E),$$

which can be rewritten, using $\Omega(df_1^\sharp, df_2^\sharp) = -\Lambda(df_1, df_2) = -\{f_1, f_2\}$ (see [8]), as

$$0 = -h_1(E.h_2 - F(E, df_2^\sharp)) + h_2(E.h_1 - F(E, df_1^\sharp)) + (F - d\omega - \Omega)(df_1^\sharp, df_2^\sharp).$$

From $\Omega(E, df_i^\sharp) = 0$, $E.h_i - d\omega(E, df_i^\sharp) = 0$ (see Lemma 2.6) it is equivalent to

$$0 = (F - d\omega - \Omega)(X_{(f_1,h_1)}, X_{(f_2,h_2)}),$$

which is satisfied for $F = \Omega + d\omega$. □

Lemma 3.2. *Let us consider the F-Lie algebroid given by the closed 2-form $F = \Omega + d\omega$. Let $(X_i, \breve{f}_i) \in \mathcal{X}(\mathbf{M}) \times C^\infty(\mathbf{M})$, $i = 1, 2$, be pairs such that X_i are infinitesimal symmetries of (ω, Ω) and $E.\breve{f}_i = -E.(\omega(X_i))$. Then the bracket $[\![,]\!]_F$ of these pairs satisfies the same conditions.*

Proof. $[X_1, X_2]$ is an infinitesimal symmetry of (ω, Ω), so it is sufficient to prove that

$$E.\big(X_1.\breve{f}_2 - X_2.\breve{f}_1 + F(X_1, X_2)\big) = -E.(\omega([X_1, X_2])),$$

which can be rewritten as

$$\begin{aligned} &X_1.(E.\breve{f}_2) - X_2.(E.\breve{f}_1) + E.(F(X_1, X_2)) \\ &\qquad = -E.(X_1.(\omega(X_2)) - X_2.(\omega(X_1) - d\omega(X_1, X_2)). \end{aligned}$$

From the condition $E.\breve{f}_i = -E.(\omega(X_i))$ this equation is satisfied if and only if

$$E.(F(X_1, X_2)) = E.(d\omega(X_1, X_2)),$$

which is satisfied for $F = \Omega + d\omega$ because $L_E\Omega = 0$ and $L_E X_i = [E, X_i] = 0$. □

Theorem 3.3. *Let us consider the F-Lie algebroid given by the closed 2-form $F = \Omega + d\omega$. Let $(X_i, \breve{f}_i) \in \mathcal{X}(\mathbf{M}) \times C^\infty(\mathbf{M})$, $i = 1, 2$, be pairs such that X_i are infinitesimal symmetries of (ω, Ω) and $E.\breve{f}_i = -E.(\omega(X_i))$. Then the sheaf mapping*

$$\mathbf{r}\colon \mathcal{X}(\mathbf{M}) \times C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M}) \times C^\infty(\mathbf{M})$$

given by

$$\mathbf{r}\colon (X, \breve{f}) \mapsto (\omega(X) + \breve{f}, \omega(X)) \tag{3.3}$$

has values in $\mathtt{LGen}(\omega, \Omega)$ and it is a Lie algebra morphism inverse to $\mathbf{s}$.

P r o o f. First we have to prove that mapping (3.3) has values in $\mathtt{LGen}(\omega,\Omega)$. Let us assume that X is an infinitesimal symmetry of (ω,Ω). By Theorem 2.5, $X = df^\sharp + hE$, where $E.f = 0$ and condition (2.5) is satisfied. It is easy to see that $h = \omega(X)$. Then

$$f = \omega(X) + \breve{f}$$

is a conserved function. So we have

$$X = d(\omega(X) + \breve{f})^\sharp + \omega(X)E \tag{3.4}$$

and the pair $(\omega(X) + \breve{f}, \omega(X))$ is in $\mathtt{LGen}(\omega,\Omega)$.

Now we have

$$\begin{aligned}[\![(X_1,\breve{f}_1);(X_2,\breve{f}_2)]\!]_F \mapsto \big(&\omega([X_1,X_2]) + X_1.\breve{f}_2 - X_2.\breve{f}_1 + \Omega(X_1,X_2)\\ &+ d\omega(X_1,X_2); \omega([X_1,X_2])\big).\end{aligned}$$

On the other hand,

$$\begin{aligned}[\![\mathbf{r}(X_1,\breve{f}_1);\mathbf{r}(X_2,\breve{f}_2)]\!] = \big(&\{\omega(X_1)+\breve{f}_1, \omega(X_2)+\breve{f}_2\}; \{\omega(X_1)+\breve{f}_1, \omega(X_2)\}\\ &- \{\omega(X_2)+\breve{f}_2, \omega(X_1)\} - d\omega(d(\omega(X_1)+\breve{f}_1)^\sharp, d(\omega(X_2)+\breve{f}_2)^\sharp)\big).\end{aligned}$$

These expressions are equal if and only if the following two equations are satisfied:

$$\begin{aligned}\omega([X_1,X_2]) + X_1.\breve{f}_2 - X_1.\breve{f}_2 + \Omega(X_1,X_2) + d\omega(X_1,X_2)&\\ = \{\omega(X_1)+\breve{f}_1, \omega(X_2)+\breve{f}_2\}&,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\omega([X_1,X_2]) = \{\omega(X_1)+\breve{f}_1, \omega(X_2)\} &- \{\omega(X_2)+\breve{f}_2, \omega(X_1)\}\\ &- d\omega(d(\omega(X_1)+\breve{f}_1)^\sharp, d(\omega(X_2)+\breve{f}_2)^\sharp).\end{aligned} \tag{3.6}$$

Using (3.4) and $(\Lambda^\sharp \otimes \Lambda^\sharp)(\Omega) = -\Lambda$, we can rewrite the left hand side of (3.5) as

$$\begin{aligned}X_1.(\omega(X_2)+\breve{f}_2) &- X_2.(\omega(X_1)+\breve{f}_1) + \Omega(X_1,X_2)\\ &= i_{d(\omega(X_1)+\breve{f}_1)^\sharp+\omega(X_1)E}\, d(\omega(X_2)+\breve{f}_2)\\ &\quad - i_{d(\omega(X_2)+\breve{f}_2)^\sharp+\omega(X_2)E}\, d(\omega(X_1)+\breve{f}_1)\\ &\quad + \Omega(d(\omega(X_1)+\breve{f}_1)^\sharp, d(\omega(X_2)+\breve{f}_2)^\sharp)\\ &= \{\omega(X_1)+\breve{f}_1, \omega(X_2)+\breve{f}_2\} - \{\omega(X_2)+\breve{f}_2, \omega(X_1)+\breve{f}_1\}\\ &\quad - \{\omega(X_1)+\breve{f}_1, \omega(X_2)+\breve{f}_2\}\\ &= \{\omega(X_1)+\breve{f}_1, \omega(X_2)+\breve{f}_2\}.\end{aligned}$$

Similarly, using $i_E d(\omega(X)) = d\omega(E, X)$, we can rewrite the left hand side of (3.6) as

$$
\begin{aligned}
X_1.(\omega(X_2)) - X_2.(\omega(X_1)) - d\omega(X_1, X_2)& \\
&= i_{d(\omega(X_1)+\breve{f}_1)^\sharp + \omega(X_1)E} d(\omega(X_2)) \\
&\quad - i_{d(\omega(X_2)+\breve{f}_2)^\sharp + \omega(X_2)E} d(\omega(X_1)) - d\omega(X_1, X_2) \\
&= \{\omega(X_1) + \breve{f}_1, \omega(X_2)\} - \{\omega(X_2) + \breve{f}_2, \omega(X_1)\} \\
&\quad - d\omega(X_1 - \omega(X_1)E, X_2 - \omega(X_2)E) \\
&= \{\omega(X_1) + \breve{f}_1, \omega(X_2)\} - \{\omega(X_2) + \breve{f}_2, \omega(X_1)\} \\
&\quad - d\omega(d(\omega(X_1) + \breve{f}_1)^\sharp, d(\omega(X_2) + \breve{f}_2)^\sharp),
\end{aligned}
$$

which proves that (3.3) is a Lie algebra morphism.

Finally, it is easy to see that $(\mathbf{r} \circ \mathbf{s})(f, h) = (f, h)$ for all $(f, h) \in \mathtt{LGen}(\omega, \Omega)$. $\square$

4. Examples

In this section we recall results obtained for structures of the classical phase space. These results were the motivation of the paper.

We assume *classical space-time* to be an oriented and time oriented 4-dimensional manifold $\mathbf{E}$ equipped with a Lorentzian metric g with signature $(1, 3)$ (see [7]). We denote by $(x^\lambda) = (x^0, x^i)$, $\lambda = 0, 1, 2, 3$, local coordinates on $\mathbf{E}$ such that ∂_0 is time-like and ∂_i are space-like. A *motion* is defined to be a 1-dimensional time-like submanifold of space-time. We define the *classical (Einsteinian) phase space* to be the open subspace $\mathcal{J}_1\mathbf{E} \subset J_1(\mathbf{E}, 1)$ consisting of all 1-jets (1st order contact elements) of motions. So elements of $\mathcal{J}_{1x}\mathbf{E}$ are classes of non-parametrized curves which have in a point $x \in \mathbf{E}$ the same tangent line lying inside the light cone. Further, $\pi^1_0 \colon \mathcal{J}_1\mathbf{E} \to \mathbf{E}$ is a fibred manifold but not an affine bundle! We have the induced coordinate chart (x^λ, x^i_0).

The metric g gives naturally the unscaled horizontal *time form*

$$\widehat{\tau} \colon \mathcal{J}_1\mathbf{E} \to T^*\mathbf{E}, \quad \widehat{\tau} = \widehat{\tau}_\lambda dx^\lambda.$$

4.1. Infinitesimal symmetries of the gravitational contact structure. The pair $(-\widehat{\tau}, \Omega^{\mathfrak{g}})$, where

$$\Omega^{\mathfrak{g}} = -d\widehat{\tau} \colon \mathcal{J}_1\mathbf{E} \to \bigwedge^2 T^*\mathcal{J}_1\mathbf{E},$$

is the contact (gravitational) regular structure on $\mathcal{J}_1\mathbf{E}$. The dual Jacobi structure is given by a pair $(-\widehat{\gamma}^{\mathfrak{g}}, \Lambda^{\mathfrak{g}})$, where $\widehat{\gamma}^{\mathfrak{g}}$ and $\Lambda^{\mathfrak{g}}$ are naturally given by the metric field (for details see [7]).

By Corollary 2.8 infinitesimal symmetries of the gravitational contact phase structure are Hamilton-Jacobi lifts of conserved functions, i.e. they are of the type $X = df^\sharp + f\widehat{\gamma}^\mathfrak{g}$, where $\widehat{\gamma}^\mathfrak{g}.f = 0$ and, moreover, $f = \widehat{\tau}(X) = \widehat{\tau}(\underline{X})$. Here $\underline{X} = T\pi^1_0(X)\colon \mathcal{J}_1\mathbf{E} \to T\mathbf{E}$ is a *generalized vector field* in the terminology of [13]. So infinitesimal symmetries are of the type

$$X = d(\widehat{\tau}(\underline{X}))^\sharp + \widehat{\tau}(\underline{X})\widehat{\gamma}^\mathfrak{g}, \tag{4.1}$$

where the following conditions are satisfied:

(1) (Projectability.) The Hamilton-Jacobi lift (4.1) projects on $\underline{X}$.

(2) (Conservation.) $\widehat{\tau}(\underline{X})$ is conserved, i.e. $\gamma^\mathfrak{g}.(\widehat{\tau}(\underline{X})) = 0$.

The following results were proved in [3].

Lemma 4.1. *A symmetric k-vector field $\overset{k}{K}$, $k \geqslant 1$, on $\mathbf{E}$ admits the generalized vector field satisfying the projectability condition. Such generalized vector fields are given by*

$$\underline{X}[\overset{k}{K}] = k\underbrace{\widehat{\tau} \lrcorner \ldots \lrcorner \widehat{\tau} \lrcorner}_{(k-1)-\text{times}}\overset{k}{K} - (k-1)\overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau})\widehat{\mathfrak{A}}\colon\ \mathcal{J}_1\mathbf{E} \to T\mathbf{E},$$

where $\widehat{\mathfrak{A}} = T\pi^1_0(\widehat{\gamma}^\mathfrak{g})$. Then we obtain the induced phase function

$$\widehat{\tau}\big(\underline{X}\,[\overset{k}{K}]\big) = \overset{k}{K}(\widehat{\tau}) = \overset{k}{K}(\widehat{\tau}, \ldots, \widehat{\tau}) = \overset{k}{K}{}^{\lambda_1\ldots\lambda_k}\widehat{\tau}_{\lambda_1}\ldots\widehat{\tau}_{\lambda_k}.$$

Lemma 4.2. *Let $\overset{0}{K}$ be a space-time function. Then $\widehat{\gamma}^\mathfrak{g}.\overset{0}{K} = 0$ if and only if $\overset{0}{K}$ is a constant. The phase function $\overset{k}{K}(\widehat{\tau})$, $k \geqslant 1$, is conserved with respect to the gravitational Reeb vector field, i.e. $\widehat{\gamma}^\mathfrak{g}.\overset{k}{K}(\widehat{\tau}) = 0$, if and only if $\overset{k}{K}$ is a Killing k-vector field.*

Theorem 4.3. *The Hamilton-Jacobi lift of a phase function*

$$K = \overset{0}{K} + \sum_{k\geqslant 1}\overset{k}{K}(\widehat{\tau}) \tag{4.2}$$

is an infinitesimal symmetry of the gravitational contact phase structure $(-\widehat{\tau}, \Omega^\mathfrak{g})$ if and only if $\overset{0}{K}$ is a constant and $\overset{k}{K}$, $k \geqslant 1$, are Killing k-vector fields. Moreover, for $k = 1$ the corresponding infinitesimal symmetry coincides with the jet flow lift $\mathcal{J}_1\overset{1}{K}$ and is projectable on space-time. For $k \geqslant 2$ the corresponding infinitesimal symmetry is hidden.

Remark 4.1. It is very well known that Killing multivector fields generate on $T^*\mathbf{E}$ *functions constant of motion* (functions constant on lifts of geodesic curves), (see, for instance, [15]). In [4] it was proved that if we consider the mapping $-\widehat{\tau}\colon \mathcal{J}_1\mathbf{E} \to T^*\mathbf{E}$, then a conserved phase function of type (4.2) is obtained as the pull-back $K = -\widehat{\tau}^*(\widetilde{K})$ of a function $\widetilde{K}$ constant of motion.

4.2. Infinitesimal symmetries of the total almost-cosymplectic-contact phase structure. Let us assume an *electromagnetic (Maxwell) field* which is a closed 2-form $\widehat{F}\colon \mathbf{E} \to \wedge^2 T^*\mathbf{E}$. Then we can consider the total phase 2-form

$$\Omega^{\mathfrak{j}} =: \Omega^{\mathfrak{g}} + \Omega^{\mathfrak{e}} = -d\widehat{\tau} + \frac{1}{2}\widehat{F},$$

and the pair $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ turns out to be an almost-cosymplectic-contact structure of the phase space, i.e. $\Omega^{\mathfrak{j}}$ is closed and $\widehat{\tau}\wedge\Omega^{\mathfrak{j}}\wedge\Omega^{\mathfrak{j}}\wedge\Omega^{\mathfrak{j}}$ is a volume form.

The dual almost-coPoisson-Jacobi pair is then given by the total Reeb vector field $\widehat{\gamma}^{\mathfrak{j}} = \widehat{\gamma}^{\mathfrak{g}} + \widehat{\gamma}^{\mathfrak{e}}$ and the total phase 2-vector $\Lambda^{\mathfrak{j}} = \Lambda^{\mathfrak{g}} + \Lambda^{\mathfrak{e}}$, where $\widehat{\gamma}^{\mathfrak{e}}$ and $\Lambda^{\mathfrak{e}}$ are $\mathbf{E}$-vertical given by g and $\widehat{F}$, see [7].

In [4] it was proved that all phase infinitesimal symmetries of the total phase structure are vector fields of the type

$$X = d(\widehat{\tau}\,(\underline{X}) + \breve{f})^{\sharp\mathfrak{j}} + \widehat{\tau}(\underline{X})\,\widehat{\gamma}^{\mathfrak{j}}$$

where $\underline{X}$ is a generalized vector field and $\breve{f} \in C^\infty(\mathbf{E})$ such that:

(1) $d\breve{f} = \underline{X} \lrcorner \widehat{F}$.
(2) (Projectability.) The vector field X projects on $\underline{X}$.
(3) (Conservation.) $\widehat{\gamma}^{\mathfrak{j}}.(\widehat{\tau}(\underline{X}) + \breve{f}) = 0$.

The projectability condition is the same as in the case of the contact gravitational structure, which follows from the fact that the fields $\widehat{\gamma}^{\mathfrak{e}}$ and $\Lambda^{\mathfrak{e}}$ are $\mathbf{E}$-vertical. So it is sufficient to describe conditions under which function (4.2), where $\breve{f} = \overset{0}{K}$, is conserved.

Theorem 4.4 ([4]). *A phase function* (4.2) *is conserved, i.e.* $\widehat{\gamma}^{\mathfrak{j}}.K = 0$, *if and only if*

$$g^{\varrho\lambda}\partial_\varrho \overset{0}{K} + \overset{1}{K}{}^{\varrho}\widehat{F}_\varrho^{\lambda} = 0, \tag{4.3}$$

$$\nabla^{(\lambda_1}\overset{k}{K}{}^{\lambda_2\dots\lambda_{k+1})} + (k+1)\overset{k+1}{K}{}^{\varrho(\lambda_1\dots\lambda_k}\widehat{F}_\varrho^{\lambda_{k+1})} = 0, \tag{4.4}$$

for $k \geqslant 1$.

Corollary 4.5. *Let us assume a special phase function* $K = \overset{0}{K} + \overset{1}{K}(\widehat{\tau})$. *Then conditions* (4.3) *and* (4.4) *are reduced to*

$$\partial_\varrho \overset{0}{K} - \overset{1}{K}{}^\sigma \widehat{F}_{\sigma\varrho} = 0, \quad \nabla^{(\lambda_1} \overset{1}{K}{}^{\lambda_2)} = 0$$

and we obtain the result of [9], *i.e.* $\overset{1}{K}$ *is a Killing vector field and* $\overset{0}{K}$ *and* $\overset{1}{K}$ *are related by the formula* $d\overset{0}{K} = \overset{1}{K} \lrcorner \widehat{F}$. *Moreover, the corresponding infinitesimal symmetry is the jet flow lift* $\mathcal{J}_1 \overset{1}{K}$, *which projects on* $\overset{1}{K}$.

Remark 4.2. Let us assume a phase function $K = \overset{k}{K}(\widehat{\tau})$, $k \geqslant 2$. Then condition (4.4) gives

$$\nabla^{(\lambda_1} \overset{k}{K}{}^{\lambda_2 \dots \lambda_{k+1})} = 0, \quad \overset{k}{K}{}^{\varrho(\lambda_1 \dots \lambda_{k-1}} \widehat{F}_\varrho^{\lambda_k)} = 0$$

and we obtain that $\overset{k}{K}$ is a *Killing-Maxwell* k-vector field. But the corresponding lift has to satisfy also condition (1), which is of the form $\underline{X}[K] \lrcorner \widehat{F} = 0$. This condition implies $\widehat{\pi} \lrcorner \widehat{F} = 0$, which implies $\widehat{F} \equiv 0$ and the structure is reduced to the gravitational case. This implies that there are no non-projectable (hidden) infinitesimal symmetries generated by Killing-Maxwell k-vector fields for $k \geqslant 2$.

So all infinitesimal symmetries of $(-\widehat{\tau}, \Omega^{\mathfrak{j}})$ are projectable and can be generated by pairs $(\underline{X}, \breve{f})$ of Killing vector fields and spacetime functions such that $d\breve{f} = \underline{X} \lrcorner \widehat{F}$. Such pairs are sections of the Lie algebroid $T\mathbf{E} \oplus \mathbb{R} \to \mathbf{E}$ with the bracket $[\![;]\!]_{\widehat{F}}$ (see Section 3 and [9]). The sections of the Lie algebroid described in Section 3 are obtained as the 1-jet flow lifts of $\underline{X}$ and the pull-backs of $\breve{f}$.

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