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A NOTE ON THE DISTRIBUTION OF ANGLES ASSOCIATED TO INDEFINITE INTEGRAL BINARY QUADRATIC FORMS

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Abstract. To each indefinite integral binary quadratic form Q, we may associate the geodesic in $\mathbb{H}$ through the roots of quadratic equation $Q(x,1)$. In this paper we study the asymptotic distribution (as discriminant tends to infinity) of the angles between these geodesics and one fixed vertical geodesic which intersects all of them.

Keywords: Weyl sum; indefinite integral binary quadratic form; real quadratic field; geodesic; asymptotic distribution

MSC 2010: 11L15, 62E20, 06B10

1. Introduction

The discriminants Δ with property that every integral binary quadratic form with discriminant Δ is primitive (i.e. its coefficients are coprime) are called *fundamental*.

Let $\Delta > 1$ be an odd fundamental discriminant, hence $\Delta \equiv 1 \pmod 4$. We consider the set $\mathcal{Q}(\Delta)$ of all indefinite integral binary quadratic forms

$$Q(x,y) = ax^2 + bxy + cy^2 \tag{1.1}$$

with discriminant Δ. To each quadratic form Q we may associate the geodesic line in $\mathbb{H}$ given explicitly by

$$\mathcal{G}_Q\colon\ z = -\frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\mathrm{e}^{\mathrm{i}\varphi}, \quad \varphi \in (0,\pi). \tag{1.2}$$

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It is a semicircle with endpoints in the roots of the quadratic equation $Q(x,1)=0$:

$$\theta_1 = \frac{-b-\sqrt{\Delta}}{2a} \quad \text{and} \quad \theta_2 = \frac{-b+\sqrt{\Delta}}{2a}. \tag{1.3}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the set of all 2×2 integral symmetric matrices: if $P \in \mathrm{SL}_2(\mathbb{Z})$, then we have the map $A \mapsto PAP^{\mathrm{T}}$ for any 2×2 integral symmetric matrix A. This induces an action of $\mathrm{SL}_2(\mathbb{Z})$ on the set $\mathcal{Q}(\Delta)$. Every form of discriminant Δ is equivalent (under the previous action) to some form (1.1) which satisfies the conditions

$$\begin{gathered} 0 < b < \sqrt{\Delta}, \\ \sqrt{\Delta} - b < 2|a| < \sqrt{\Delta} + b, \end{gathered} \tag{1.4}$$

and such a form is called *reduced*. We remark that the representative may be not unique, but every class has a finite number of representatives because the number of reduced forms is finite. For more details see [1], Section 3.1.

Because conditions (1.4) depend only on $|a|$, without loss of generality we can assume that $a > 0$, which implies $c < 0$. Let $\mathcal{H}(\Delta)$ be the set of all lattice points $(a,b,c) \in \mathbb{Z}^3$ on the one-sheeted hyperboloid $b^2 - 4ac = \Delta$ which satisfy these conditions. We will identify points $(a,b,c) \in \mathcal{H}(\Delta)$ with the corresponding quadratic form $Q(x,y) = ax^2 + bxy + cy^2$. Lists of elements $\mathcal{H}(\Delta)$ for small discriminants Δ can be found in [1], page 30. Conditions (1.4) are equivalent to

$$\theta_1 < -1 \quad \text{and} \quad 0 < \theta_2 < 1. \tag{1.5}$$

By Dirichlet class number formula, we have

$$|\mathcal{H}(\Delta)| \sim \frac{6}{\pi^2} \log 2\sqrt{\Delta} L(1, \chi_\Delta), \tag{1.6}$$

as $\Delta \to \infty$ over odd fundamental discriminants. The lower bound is $L(1,\chi_\Delta) \gg 1/\Delta^\varepsilon$ for any $\varepsilon > 0$; Siegel's theorem implies $|\mathcal{H}(\Delta)| \gg \Delta^{1/2-\varepsilon}$. For this reason, various statistical questions are possible in the limit as $\Delta \to \infty$.

Let $t \in [0,1]$ be a fixed parameter. Vertical geodesic $p_t\colon \Re z = -t$ intersects all geodesics $\mathcal{G}_Q$, for $Q \in \mathcal{H}(\Delta)$.

For $\alpha \in [0,\pi]$ we denote

$$\mathcal{H}(\Delta; t, \alpha) = \{Q \in \mathcal{H}(\Delta)\colon \sphericalangle(p_t, \mathcal{G}_Q) \leqslant \alpha\}, \tag{1.7}$$

where $\sphericalangle(p_t, \mathcal{G}_Q)$ is the angle from p_t to tangent on the semicircle $\mathcal{G}_Q$ in the counterclockwise direction. We can see all geodesics and intersection angles in the case $\Delta = 41$ in Figure 1. In this paper we study the *distribution function* of this angle.

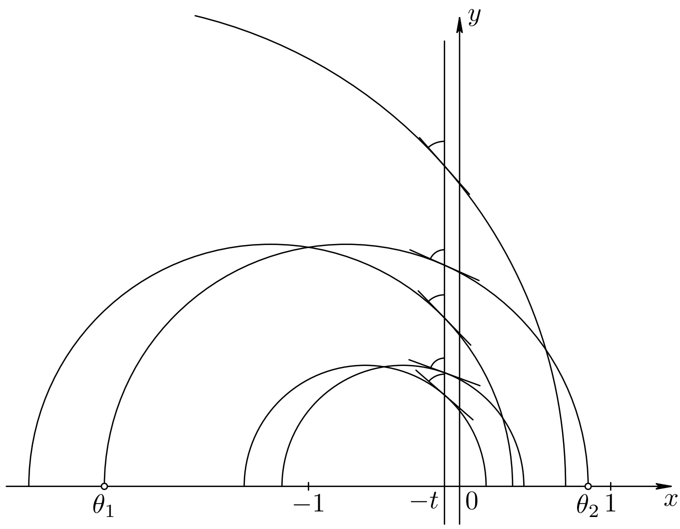

Figure 1. For $\Delta = 41$ we have five triples for $(a,b,c) \in \mathcal{H}(41)$: $(2,3,-4)$, $(4,3,-2)$, $(1,5,-4)$, $(2,5,-2)$ and $(4,5,-1)$. Line p_t intersects all the corresponding geodesics.

For a finite Δ the distribution function $|\mathcal{H}(\Delta;t,\alpha)|/|\mathcal{H}(\Delta)|$ is a step function. However, its limit function is continuously differentiable, precisely:

Theorem 1.1. *Let $t \in [0,1]$. Then, as $\Delta \to \infty$ over odd fundamental discriminants, we have*

$$(1.8)\qquad \begin{cases} \dfrac{|\mathcal{H}(\Delta;t,\alpha)|}{|\mathcal{H}(\Delta)|} \sim \dfrac{\log\frac{1+t}{t}}{2\log 2}(1-\cos\alpha) & \text{if } \alpha \in [0, \arccos(1-2t)), \\ \dfrac{|\mathcal{H}(\Delta;t,\alpha)|}{|\mathcal{H}(\Delta)|} \sim 1 - \dfrac{(1-\cos\alpha)\log\frac{1-\cos\alpha}{1+t} + (1+\cos\alpha)\log\frac{1+\cos\alpha}{1-t}}{2\log 2} & \\ \qquad\qquad & \text{if } \alpha \in [\arccos(1-2t), \arccos(-t)), \\ \dfrac{|\mathcal{H}(\Delta;t,\alpha)|}{|\mathcal{H}(\Delta)|} \sim 1 & \text{if } \alpha \in [\arccos(-t), \pi]. \end{cases}$$

In other words, the angle is equidistributed with respect to the measure

$$(1.9)\qquad \mathrm{d}\mu x = \begin{cases} \dfrac{\log\frac{1+t}{t}}{2\log 2}\sin x\,\mathrm{d}x & \text{if } x \in [0, \arccos(1-2t)), \\ \dfrac{1}{2\log 2}\sin x \log\dfrac{(1+t)(1+\cos x)}{(1-t)(1-\cos x)}\,\mathrm{d}x & \\ \qquad & \text{if } x \in [\arccos(1-2t), \arccos(-t)), \\ 0 & \text{if } x \in [\arccos(-t), \pi]. \end{cases}$$

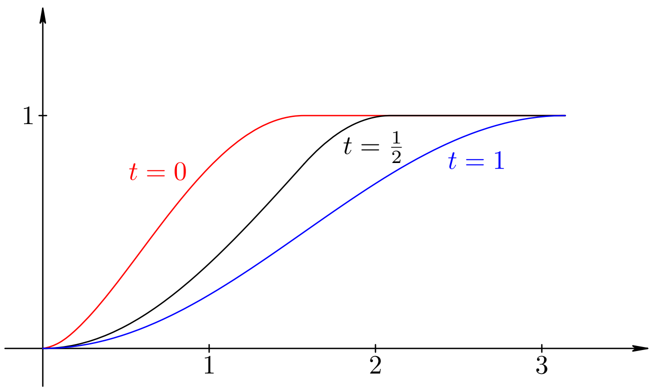

Figure 2. Graphic of the distribution function $\lim_{\Delta\to\infty} |\mathcal{H}(\Delta; t, \alpha)|/|\mathcal{H}(\Delta)|$.

Figure 2 illustrates distribution of angles for some typical values of parameter t.

Duke, Friedlander and Iwaniec considered the distribution of the negative root θ_1 (see Section 2 or [2]). Every point $(a, b, c) \in \mathcal{H}(\Delta)$ gives rise to an ideal $a\mathbb{Z} + \frac{1}{2}(b + \sqrt{\Delta})\mathbb{Z}$ of the ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{\Delta})}$ of the number field $\mathbb{Q}(\sqrt{\Delta})$. Hence, authors of [2] found in a sense the distribution of the generators of ideals in $\mathcal{O}_{\mathbb{Q}(\sqrt{\Delta})}$. Theorem 1.1 gives in a sense the distribution of angle associated to the pair of ideals

$$\mathfrak{a}_1 = a\mathbb{Z} + \frac{-b + \sqrt{\Delta}}{2}\mathbb{Z} \quad \text{and} \quad \mathfrak{a}_2 = a\mathbb{Z} + \frac{b + \sqrt{\Delta}}{2}\mathbb{Z}, \tag{1.10}$$

which are extensions of the ideal $a\mathbb{Z} \triangleleft \mathbb{Z}$ that is $a\mathbb{Z} = \mathfrak{a}_1\mathfrak{a}_2$. One may say that this is the angle associated to ideals $a\mathbb{Z}$, for $a \in \mathbb{Z}$ such that $\Delta/4a = 1$.

Theorem 1.1 is a corollary of Duke-Friedlander-Iwaniec theorem and it will be proven in Section 3. Next, we give some variations (corollaries) of this theorem in Section 4.

2. Weyl sums and Duke-Friedlander-Iwaniec theorem

The main ingredient of our proof is Duke-Friedlander-Iwaniec theorem for *Weyl sums* for quadratic roots ([2], Theorem 1.1). We present a brief summary of their results.

As is well known, equidistribution results are based on the estimations for the Weyl sums. Let X be a topological space with probablistic measure. Weyl sums are the sums $\sum_x f(x)$, where f is the eigenfunction from the spectral expansion of $\mathcal{L}^2(X)$.

Specially, when the space X is a circle $\mathbb{R}/\mathbb{Z}$, the spectrum consists of eigenfunctions $e(hx)$, $h \in \mathbb{Z}$, where $e(z) = \mathrm{e}^{2\pi i z}$. Weyl's criterion for equidistribution modulo 1 can be found in [4], Theorem 11.1.5.

Authors of [2] considered the Weyl sum over roots of quadratic congruences:

$$W_h(\Delta; c) = \sum_{\substack{b \pmod{c} \\ b^2 \equiv \Delta \pmod{c}}} e\Big(h\frac{b}{c}\Big). \tag{2.1}$$

These sums have only a few terms, bounded by the divisor function, so there is not much room to cancellation. For applications there is a lot of interest in bounds for sums of these sums as c varies over multiples of some modulus q:

$$\mathcal{W}_h(\Delta) = \sum_{c \equiv 0 \pmod{q}} f(c) W_h(\Delta; c). \tag{2.2}$$

Theorem 2.1 (Duke-Friedlander-Iwaniec, [2], Theorem 1.1)**.** *Let $h \geqslant 1$, $q \geqslant 1$ and let Δ be a positive odd fundamental discriminant. Let $f(y)$ be a smooth function supported on $Y \leqslant y \leqslant 2Y$ with $Y \geqslant 1$, such that*

$$|f(y)| \leqslant 1, \quad y^2|f''(y)| \leqslant 1. \tag{2.3}$$

Then

$$W_h(\Delta) \ll h^{1/4}(Y + \sqrt{\Delta})^{3/4}\Delta^{1/8-1/1331} \tag{2.4}$$

as $\Delta \to \infty$, where the implied constant is absolute.

This theorem is obtained using the spectral theory of automorphic forms. For related results for negative discriminants see [3], Section 21.

For a smooth function W compactly supported on $\mathbb{R}^+ \times \mathbb{R}^+$ authors of [2] considered the sum

$$\mathcal{W}(\Delta) = \sum_{b^2-4ac=\Delta} W\Big(\frac{a}{\sqrt{\Delta}}, \frac{b}{\sqrt{\Delta}}\Big), \tag{2.5}$$

and using Theorem 2.1, proved ([2], Sections 12 and 13) that

$$\mathcal{W}(\Delta) = RW\sqrt{\Delta} + \mathcal{O}(\Delta^{1/2-1/1331}) \tag{2.6}$$

as $\Delta \to \infty$ over odd fundamental discriminants, where

$$R = \frac{3}{\pi^2}L(1,\chi_\Delta) \quad \text{and} \quad W = \iint_{\mathbb{R}^+\times\mathbb{R}^+} W(u,v)\frac{\mathrm{d}u\,\mathrm{d}v}{u}. \tag{2.7}$$

Using formula (2.6) we will prove Theorem 1.1 in the next section.

In the mentioned paper, authors gave four geometric applications. One of them is the distribution of negative roots of indefinite integral binary quadratic forms (with the same assumptions as ours). For $\lambda > 1$, authors denoted by $\mathcal{H}(\Delta;\lambda)$ a subset of points in $\mathcal{H}(\Delta)$ such that the absolute values of negative roots of the corresponding forms are at most λ, i.e.

$$\mathcal{H}(\Delta;\lambda) = \Big\{(a,b,c) \in \mathcal{H}(\Delta)\colon\ \frac{b+\sqrt{\Delta}}{2a} \leqslant \lambda\Big\}. \tag{2.8}$$

In Theorem 1.4. in [2] (which is essentially a corollary of (2.6)), authors established the asymptotics

$$\frac{|\mathcal{H}(\Delta;\lambda)|}{|\mathcal{H}(\Delta)|} \sim \frac{1}{\log 2}\log\frac{2\lambda}{\lambda+1} \tag{2.9}$$

as $\Delta \to \infty$ over odd fundamental discriminants. In other words, the first root (1.3) is equidistributed with respect to the measure

$$\mathrm{d}\mu x = \frac{1}{\log 2}\frac{\mathrm{d}x}{x(x+1)}.$$

3. Proof of Theorem 1.1

From elementary geometry (see Figure 3) it is known that the angle between geodesic $\mathcal{G}_Q\colon\ -b/2a + \sqrt{\Delta}\mathrm{e}^{\mathrm{i}\varphi}/2a$ and line p_t is equal to the argument φ_0 of the intersection point of $\mathcal{G}_Q$ and p_t. Next, we can calculate that $\cos\varphi_0 = (b-2at)/\sqrt{\Delta}$.

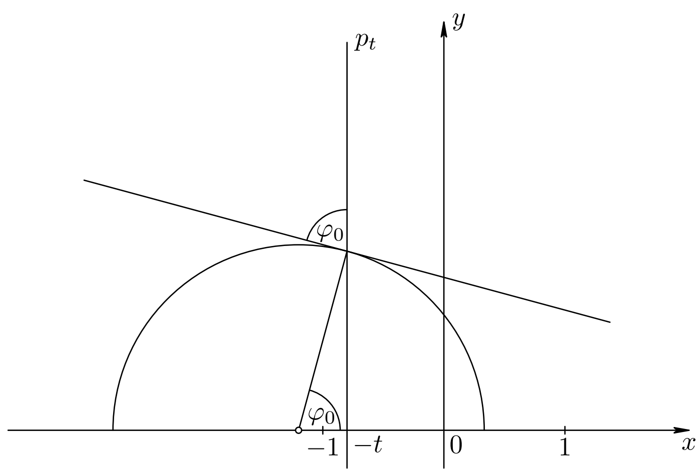

Figure 3. Angle between geodesic and line p_t.

It is clear that $|\mathcal{H}(\Delta; t, \alpha)|$ counts lattice points on a hyperboloid which satisfy conditions (1.4) and the additional condition $(b-2at)/\sqrt{\Delta} \geqslant \cos\alpha$ (because cosine is a decreasing function on the segment $[0, \pi]$). A natural attempt to apply (2.6) to the characteristic function of this set would not give result because this function is not compactly supported in $\mathbb{R}^+ \times \mathbb{R}^+$.

Consequently, we will replace the set $\mathcal{H}(\Delta; t, \alpha)$ by its complement

$$\widehat{\mathcal{H}}(\Delta; t, \alpha) = \mathcal{H}(\Delta) \setminus \mathcal{H}(\Delta; t, \alpha). \tag{3.1}$$

Analogously, $|\widehat{\mathcal{H}}(\Delta; t, \alpha)|$ counts lattice points on a hyperboloid which satisfy conditions (1.4) and the additional condition $(b-2at)/\sqrt{\Delta} < \cos\alpha$.

Denoting $u = a/\sqrt{\Delta}$, $v = b/\sqrt{\Delta}$ these conditions can be expressed as

$$0 < 1 - v < 2u < 1 + v, \quad v - 2ut < \cos\alpha \tag{3.2}$$

or

$$|2u - 1| < v < \min\{1, 2ut + \cos\alpha\}. \tag{3.3}$$

Note that the set (3.3) is not empty iff

$$\frac{1-\cos\alpha}{2(1+t)} < u < \min\left\{1, \frac{1+\cos\alpha}{2(1-t)}\right\} \quad \text{and} \quad \cos\alpha > -t. \tag{3.4}$$

For $\alpha \geqslant \arccos(-t)$, it is clear that $|\mathcal{H}(\Delta; t, \alpha)|/|\mathcal{H}(\Delta)| = 1$, because $\widehat{\mathcal{H}}(\Delta; t, \alpha) = \emptyset$.

The number $|\widehat{\mathcal{H}}(\Delta; t, \alpha)|$ is exactly the Weyl sum for $W = \chi_\Omega$, where Ω is the domain in $\mathbb{R}^2$ defined by conditions (3.2). Function χ_Ω is supported in $\mathbb{R}^+ \times \mathbb{R}^+$ for $\alpha > 0$. (We know that $|\mathcal{H}(\Delta; t, 0)|/|\mathcal{H}(\Delta)| = 0$ for all t, because $\mathcal{H}(\Delta; t, 0) = \emptyset$.) Function χ_Ω is not smooth and we cannot apply (2.6) directly. However, we can smooth it out by modifying (increasing and decreasing) it within a set of asymptotically shrinking measure. By positivity, this gives us an upper and a lower bound for $|\mathcal{H}(\Delta; t, \alpha)|$, both asymptoticaly equal. Hence the formula (2.6) is justified for characteristic function χ_Ω. From $|\widehat{\mathcal{H}}(\Delta; t, \alpha)| \sim RW\sqrt{\Delta}$ and (1.6) it follows that

$$\frac{|\mathcal{H}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} \sim 1 - \frac{W}{2\log 2}, \tag{3.5}$$

as $\Delta \to \infty$ over odd fundamental discriminants, where

$$W = \iint\limits_{\mathbb{R}^+ \times \mathbb{R}^+} \chi_\Omega(u, v) \frac{\mathrm{d}u\,\mathrm{d}v}{u}.$$

For $\alpha \in [\arccos(1-2t), \arccos(-t))$ we have

$$\begin{aligned}
W &= \int_{(1-\cos\alpha)/2(1+t)}^{(1+\cos\alpha)/2(1-t)} (\min\{1, \cos\alpha + 2ut\} - |2u-1|)\frac{\mathrm{d}u}{u} \\
&= \int_{(1-\cos\alpha)/2(1+t)}^{1/2} (\cos\alpha + 2ut + 2u - 1)\frac{\mathrm{d}u}{u} \\
&\quad + \int_{1/2}^{(1+\cos\alpha)/2(1-t)} (\cos\alpha + 2ut - 2u + 1)\frac{\mathrm{d}u}{u} \\
&= 2(1+t)\frac{t+\cos\alpha}{2(1+t)} + (\cos\alpha - 1)\log\frac{1+t}{1-\cos\alpha} + 2(t-1)\frac{t+\cos\alpha}{2(1-t)} \\
&\quad + (\cos\alpha + 1)\log\frac{1+\cos\alpha}{1-t} \\
&= (1-\cos\alpha)\log\frac{1-\cos\alpha}{1+t} + (1+\cos\alpha)\log\frac{1+\cos\alpha}{1-t}.
\end{aligned}$$

Similarly, for $\alpha < \arccos(1-2t)$ we have

$$\begin{aligned}
W &= \int_{(1-\cos\alpha)/2(1+t)}^{1} (\min\{1, \cos\alpha + 2ut\} - |2u-1|)\frac{\mathrm{d}u}{u} \\
&= \int_{(1-\cos\alpha)/2(1+t)}^{(1-\cos\alpha)/2t} (\cos\alpha + 2ut)\frac{\mathrm{d}u}{u} + \int_{(1-\cos\alpha)/2t}^{1} \frac{\mathrm{d}u}{u} \\
&\quad + \int_{(1-\cos\alpha)/2(1+t)}^{1/2} (2u-1)\frac{\mathrm{d}u}{u} + \int_{1/2}^{1} (-2u+1)\frac{\mathrm{d}u}{u} \\
&= \cos\alpha\log\frac{1+t}{t} + 2t\frac{1+\cos\alpha}{2(1+t)t} + \log\frac{2t}{1-\cos\alpha} + 2\frac{t+\cos\alpha}{2(1+t)} \\
&\quad - \log\frac{1+t}{1-\cos\alpha} - 2\cdot\frac{1}{2} + \log 2 \\
&= 2\log 2 - (1-\cos\alpha)\log\frac{1+t}{t}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

4. Some variations

We could consider the absolute angle (smaller of the two supplementary angles) between the geodesic $\mathcal{G}_Q$ and line p_t. Let $\widetilde{\mathcal{H}}(\Delta; t, \alpha)$ be a set of triples in $\mathcal{H}(\Delta)$ such that the absolute angle is at most α. We can determine the distribution of the absolute angle from the relation

$$\frac{|\widetilde{\mathcal{H}}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} = \frac{|\mathcal{H}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} + 1 - \frac{|\mathcal{H}(\Delta; t, \pi - \alpha)|}{|\mathcal{H}(\Delta)|} \tag{4.1}$$

and Theorem 1.1. There are several cases according to the parameter t. For example, for $t \in \left[0, \frac{1}{3}\right)$ we have

$$
(4.2)\qquad \begin{cases}
\dfrac{|\widetilde{\mathcal{H}}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} \sim \dfrac{\log \frac{1+t}{t}}{2\log 2}(1-\cos\alpha) & \text{if } \alpha \in [0, \arccos(1-2t)),\\
\dfrac{|\widetilde{\mathcal{H}}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} \sim 1 - \dfrac{(1-\cos\alpha)\log\frac{1-\cos\alpha}{1+t} + (1+\cos\alpha)\log\frac{1+\cos\alpha}{1-t}}{2\log 2} & \\
\qquad\qquad \text{if } \alpha \in [\arccos(1-2t), \arccos t], & \\
\dfrac{|\widetilde{\mathcal{H}}(\Delta; t, \alpha)|}{|\mathcal{H}(\Delta)|} \sim 1 - \dfrac{\log\frac{1+t}{1-t}}{\log 2}\cos\alpha & \text{if } \alpha \in (\arccos t, \frac{\pi}{2}],
\end{cases}
$$

as $\Delta \to \infty$ over odd fundamental discriminants.

Of course, the same results are valid for $a < 0$, line $\Re z = t$ and angle in clockwise direction. Similarly, for $a \in \mathbb{Z} \setminus \{0\}$, all geodesics intersect the line $\Re z = 0$. Let $\mathfrak{H}(\Delta)$ be a set of points on hyperboloid $b^2 - 4ac = \Delta$ which satisfy (1.4) (without $a > 0$) and $\mathfrak{H}(\Delta; \alpha)$ a subset of triples in $\mathfrak{H}(\Delta)$ such that the angle is at most α. As $\Delta \to \infty$ over odd fundamental discriminants we have

$$
(4.3)\qquad \frac{|\mathfrak{H}(\Delta;\alpha)|}{|\mathfrak{H}(\Delta)|} \sim \frac{1}{2} + \operatorname{sgn}\Big(\alpha - \frac{\pi}{2}\Big) \times \frac{(1-\cos\alpha)\log(1-\cos\alpha) + (1+\cos\alpha)\log(1+\cos\alpha)}{4\log 2}.
$$

Final remark. A natural generalization of our problem is a distribution function of the angle between the geodesics $\mathcal{G}_Q$, $Q \in \mathcal{Q}(\Delta)$, and an arbitrary geodesic $\mathcal{G}$ in $\mathbb{H}$ which intersects all geodesics $\mathcal{G}_Q$. When the geodesic $\mathcal{G}$ is a semicircle $C_{k,r}\colon\ k + re^{i\psi}$, the intersection angle (from $C_{k,r}$ to $\mathcal{G}_Q$ in counter-clockwise direction) is equal to $\varphi_0 + \pi - \psi_0$, where φ_0 and ψ_0 are the arguments of the intersection point. One can derive

$$
\cos(\varphi_0 + \pi - \psi_0) = \frac{(b+2ka)^2 - \Delta - 4r^2a^2}{4ra\sqrt{\Delta}}.
$$

By the same procedure as in the proof of Theorem 1.1, we can derive the distribution of the mentioned angle. The only difference is that the characteristic function χ_Ω may be not compactly supported in $\mathbb{R}^+ \times \mathbb{R}^+$. However, then the procedure works for the sequence of functions $\chi_{\Omega_n} \uparrow \chi_\Omega$, where $\Omega_n = \Omega \cap ([1/n, \infty) \times \mathbb{R})$. Unfortunately, in this case, the obtained formula is very complicated.

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