BIFURCATION OF PERIODIC SOLUTIONS TO NONLINEAR MEASURE DIFFERENTIAL EQUATIONS

Maria Carolina Mesquita, São Carlos, Milan Tvrdý, Prague

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Dedicated to the memory of Jaroslav Kurzweil

Abstract. The paper is devoted to the periodic bifurcation problems for generalizations of ordinary differential systems. The bifurcation is understood in the static sense of Krasnoselskiĭ and Zabreĭko. First, the conditions necessary for the given point to be bifurcation point for non autonomous generalized ordinary differential equations (based on the Kurzweil gauge type generalized integral) are proved. Then, as the main contribution, analogous results are obtained also for the nonlinear non autonomous measure differential equations considered in the sense of distributions. To this aim their relationship to Kurzweil's generalized differential equations is disclosed. Although the measure differential equations turned out to be special cases of those Kurzweil's equations, the proofs of the main results of the paper are by no means the straightforward consequences of the analogous results for generalized differential equations. Essentially they rely on the theory of the Kurzweil-Stieltjes integration. It is worth noting that as the systems studied in the paper encompass many types of equations such as impulsive differential equations, ordinary differential equations, dynamic equations on time scales etc., the results of the paper offer applications to rather wide scale of practical problems. Two illustrating examples are included, as well.

Keywords: periodic solution; bifurcation; Kurzweil integral; Kurzweil-Stieltjes integral; generalized differential equation; measure differential equation; distributional differential equation

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1. Introduction

In this article we consider bifurcation properties of periodic solutions of the non autonomous measure differential system

$$\mathrm{D}x = f(\lambda, x, t) + g(x, t) \cdot \mathrm{D}u, \tag{1.1}$$

where D stands for the distributional derivatives and λ is a parameter. To this end, an important tool are generalized ordinary differential equations introduced in the middle of the 1950s by Kurzweil in [23], [24]. Since then, many authors have dealt with the potentialities of this theory, see e.g. [4], [25], [32], [43] and the references therein. The concept of measure differential equations arose more or less together with the concepts of impulse systems or distributional differential equations. They generally try to describe some physical or biological problems, such as heartbeat, blood flow, pulse/frequency modulated systems, and/or models for biological neural networks. In these models, derivatives are understood in the sense of distributions and the solutions are generally discontinuous, but not too bad from another point of view, i.e., they are usually regulated or have bounded variation. Early results were summarized e.g. in monographs, see [3], [34], [39] and the references therein. The study of such problems has also been motivated by some models developed in control theory, in which it turned out that measures can be very suitable controls, cf. e.g. [31]. Moreover, differential equations with measure also appear in non-smooth mechanics, cf. [5]. More recent references are e.g. [6], [7], [36], [40] and many others.

The study of the bifurcation phenomena was initiated by Krasnosel'skiĭ and Zabreiko, cf. [21], Section 56. Since then it has been considered by many authors, cf. e.g. [1], [2], [22], [37], and the references therein. In general, one can observe two rather different approaches. The dynamical one is a part of dynamical systems theory and considers mainly the situations when a solution of an evolutionary equation changes its stability properties with newly generated solutions or when some changes of phase portraits like saddle-node bifurcations or pitchfork bifurcations occur. Naturally, in this setting, the authors restrict themselves mostly to autonomous systems. On the other hand, in static bifurcation (or branching) theory, the existence of bifurcating (or branching) points for equations of the abstract form $F(x;\lambda) = 0$ with a scalar parameter $\lambda \in \Lambda$ is considered. Assuming that $F(x_0, \lambda) = 0$ for all $\lambda \in \Lambda$, the element of (x_0, λ_0) of the family $\{(x_0, \lambda); \lambda \in \Lambda\}$ is said to be the bifurcation point of this equation if there are sequences $\{x_n\}$ and $\{\lambda_n\}$ such that $F(x_n, \lambda_n) = 0$ for all n, $x_n \to x_0$ and $\lambda_n \to \lambda_0$, while $x_0 \notin \{x_n\}$. In other words, cf. [21], Section 56.1, (x_0, λ_0) is a bifurcation point of the given equation if for each $\varepsilon > 0$ there is a $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that the equation has at least one solution x

different from x_0 and such that $\|x - x_0\| < \varepsilon$. In practical problems, bifurcation points can be associated with such notions like critical weight in stability problems, critical values of the parameter in problems on the generation of periodic solutions, etc. In general, the non autonomous systems are included.

In this paper we follow the static type setting. In a sense, we continue the research done in [12], where the authors introduced the concept of bifurcation point with respect to the trivial solution of the periodic problem for the non autonomous generalized ordinary differential equations in the sense of Kurzweil. By means of the coincidence degree theory, they established conditions sufficient for the existence of such a bifurcation point, cf. [12], Theorem 5.6. Similar questions have been already studied in the thesis (see [30]) by the first author.

The paper is divided as follows: In Section 2, we recall the Kurzweil's concept of generalized ordinary differential equations written as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(\lambda, x, t) \tag{1.2}$$

together with some of the basic properties of the Kurzweil integral that are needed later.

Section 3 is devoted to the bifurcation theory for the periodic problem for generalized ordinary differential equations. One of the crucial assumptions is that there is a function x_0 which is a solution to (1.2) for all $\lambda \in \Lambda$. Having a proper operator representation for (1.2), we can state Theorem 3.5 giving sufficient conditions for the existence of a bifurcation point (x_0, λ_0). Its proof is an easy modification of that of Theorem 5.6 in [12]. Then, in Theorem 3.11, we present conditions necessary for the existence of a bifurcation point of the T-periodic problem for (1.2). Main argument is the abstract Implicit Function Theorem. Similar questions have been already studied in the thesis (see [30]) by the first author. Finally, an alternative version of Theorem 3.11 of the Fredholm alternative type is given, cf. Theorem 3.13.

In Section 4, we are clarifying the setting of measure differential equations (1.1) and we show that under some assumptions, there is a correspondence between its solutions and solutions of a Stieltjes integral equation of the form

$$x(t) = x(0) + \int_0^t f(\lambda, x(s), s)\,\mathrm{d}s + \int_0^t g(x(s), s)\,\mathrm{d}u(s) \quad \text{for } t \in [0, T], \tag{1.3}$$

where the integrals stand for the Kurzweil-Stieltjes ones, see Theorem 4.8. On the other hand, it is also shown that integral equation (1.3) is a special case of generalized ODE (1.2).

Our main results are located in the final Section 5. In particular, Theorem 5.8 provides the conditions necessary for the given couple (x_0, λ_0) to be a periodic bifurcation point of (1.1). The proof relies on the already mentioned results from the previous sections. Moreover, new result on exchanging order in iterated integrals presented by Lemma 5.4 has been utilized. The theory is illustrated by two examples. The former one, Example 5.10, is very simple first order scalar equation with one impulse. It was already treated in [12], where the existence of the bifurcation point was exhibited. Here, we are able to state also the uniqueness of this bifurcation point. The latter one is more sophisticated singular non autonomous equation of the second order with rather artificially added linear part and one impulse. Nevertheless, it is still related to the Liebau valveless pumping phenomena described e.g. in [35] and recently rather intensively studied in the theory of singular problems, cf. e.g. [8], [9], [45]. On the other hand, this example provides a nontrivial application of the recent results by Lomtatidze (see [27]) and/or Hakl and Torres (see [15]) giving conditions ensuring the non resonance of second order linear periodic problems.

2. Preliminaries (Kurzweil integral and generalized ODEs)

Among our tools an exceptional role is played by the Kurzweil integral and its special case, Kurzweil-Stieltjes integral. This kind of integral has been introduced by Kurzweil in the middle of the fifties, cf. [23], [24]. In this section, we summarize some of its basic properties needed later.

Throughout the paper, the symbol X stands for a Banach space equipped with the norm $\|\cdot\|_X$. Usually we restrict ourselves to the cases $X = \mathbb{R}^n$ or $X = \mathcal{L}(\mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^n)$ is the space of real $n \times n$-matrices equipped with the norm

$$\|A\|_{n\times n} = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^{n} |a_{i,j}| \quad \text{for } A = (a_{i,j})_{i,j\in\{1,\dots,n\}} \in \mathcal{L}(\mathbb{R}^n)$$

and $\mathbb{R}^n$ is the space of real $n \times 1$-matrices equipped with the norm

$$\|x\|_n = \sum_{i=1}^{n} |x_i| \quad \text{for } \; x = (x_i)_{i\in\{1,\dots,n\}} \in \mathbb{R}^n.$$

The function $x\colon [a,b] \to X$ is *regulated* if the lateral limits

$$x(t-) = \lim_{\tau\to t-} x(\tau) \quad \text{and} \quad x(s+) = \lim_{\tau\to s+} x(\tau)$$

exist for all $t \in (a,b]$ and $s \in [a,b)$. The space of functions $x\colon [a,b] \to X$ which are regulated on $[a,b]$ will be denoted as $G([a,b];X)$. As usual, $\Delta^+x(t) = x(t+) - x(t)$ and $\Delta^-x(t) = x(t) - x(t-)$ whenever the expressions on the right sides have sense. It is well known that, when equipped with the supremal norm $\|x\|_\infty = \sup\limits_{t\in[a,b]} \|x(t)\|_X$, $G([a,b];X)$ is a Banach space, see e.g. [18]. As usual, the symbol $\operatorname{var}_a^b f$ stands for the variation of the function $f\colon [a,b]\to X$ on $[a,b]$ and $\mathrm{BV}([a,b];X)$ is the space of functions $f\colon [a,b]\to X$ having a bounded variation on $[a,b]$. $\mathrm{BV}([a,b];X)$ is a Banach space with respect to the norm $\|f\|_{\mathrm{BV}} = \|f(a)\|_X + \operatorname{var}_a^b f$.

In this paper, by an integral we mean the integral introduced by Kurzweil in [23]. Its definition relies on the notions of gauges and tagged partitions fine with respect to the gauges:

Let $[a,b]$ be a bounded closed interval. Finite collections of point-interval pairs $P = (\tau_j,[\sigma_{j-1},\sigma_j])_{j=1}^{\nu(P)}$ such that $a=\sigma_0\leqslant\sigma_1\leqslant\ldots\leqslant\sigma_{\nu(P)}=b$ and $\tau_j\in[\sigma_{j-1},\sigma_j]$ for $j\in\{1,\ldots,n\}$ are called *tagged partitions* of $[a,b]$. Furthermore, any positive function $\delta\colon [a,b]\to(0,\infty)$ is called a *gauge* on $[a,b]$. Given a gauge δ on $[a,b]$, the partition $P=(\tau_j,[\sigma_{j-1},\sigma_j])_{j=1}^{\nu(P)}$ is called δ-*fine* if

$$[\sigma_{j-1},\sigma_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))\quad\text{for all } j\in\{1,2,\ldots,\nu(P)\}.$$

Recall that by Cousin Lemma in [10] (see also e.g. [43], Lemma 1.4 or [32], Lemma 6.2.3) for each gauge δ on $[a,b]$ there always exists a δ-fine tagged partition of $[a,b]$.

Definition 2.1. Let $-\infty<a<b<\infty$ and let X be a Banach space. Then the function $U\colon [a,b]\times[a,b]\to X$ is said to be *Kurzweil integrable* on $[a,b]$ if there is an $I\in X$ such that for every $\varepsilon>0$ we can find a gauge δ on $[a,b]$ such that

$$\Bigl\|\sum_{j=1}^{\nu(P)}[U(\tau_j,\sigma_j)-U(\tau_j,\sigma_{j-1})]-I\Bigr\|_X<\varepsilon$$

holds for every δ-fine tagged partition $P=(\tau_j,[\sigma_{j-1},\sigma_j])_{j=1}^{\nu(P)}$ of $[a,b]$.

In such a case, I is said to be the Kurzweil integral of U over $[a,b]$ and we write

$$I=\int_a^b \mathrm{D}U(\tau,\sigma).$$

If the integral $\int_a^b \mathrm{D}U(\tau,\sigma)$ has sense, we put

$$\int_b^a \mathrm{D}U(\tau,\sigma) = -\int_a^b \mathrm{D}U(\tau,\sigma).$$

Furthermore,

$$\int_a^b \mathrm{D}U(\tau,\sigma)=0\quad\text{if } a=b.$$

Remark 2.2.

(i) If $U(\tau,\sigma) = G(\tau)\,H(\sigma)$, where $G\colon\ [a,b]\to\mathcal{L}(X)$ and $H\colon\ [a,b]\to X$, then the integral $\int_a^b \mathrm{D}U(\tau,\sigma)$ reduces to the Kurzweil-Stieltjes integral $\int_a^b G\,\mathrm{d}H$. Similarly, if $U(\tau,\sigma) = H(\sigma)G(\tau)$, where $H\colon\ [a,b]\to\mathcal{L}(X)$ and $G\colon\ [a,b]\to X$, then $\int_a^b \mathrm{D}U(\tau,\sigma) = \int_a^b \mathrm{d}HG$. Both these cases were considered in details in [33]. For $X=\mathbb{R}$, see also [32]. Finally, if $H(\sigma)\equiv\sigma$, the integral is known as the Henstock-Kurzweil integral.

(ii) Recall that the notation $\int_a^b \mathrm{D}U(\tau,\sigma)$ for the Kurzweil integral is entirely symbolic. The letters τ and σ do not mean the actual variables of the function U, but, roughly speaking, the former one refers to tags, while the second one to interval divisions. Kurzweil originally wrote $\int_a^b \mathrm{D}U(\tau,t)$. However, we decided to leave this tradition as we found that it is helpful to “free” the letter t for other purposes.

The first part of the following assertion follows from [25], Corollary 14.18. The second one follows directly from the definition of the Kurzweil integral.

Lemma 2.3. *Let $U\colon\ [a,b]\times[a,b]\to X$ be Kurzweil integrable and regulated in the second variable on $[a,b]$ and*

$$v(t) = \int_a^t \mathrm{D}U(\tau,\sigma) \quad \text{for } t\in[a,b].$$

Then v is regulated on $[a,b]$,

$$\Delta^- v(t) = U(t,t) - U(t,t-) \quad \text{if } t\in[a,b)$$

and

$$\Delta^+ v(t) = U(t,t+) - U(t,t) \quad \text{if } t\in(a,b].$$

Moreover, if there are functions $f\colon\ [a,b]\to\mathbb{R}$ regulated on $[a,b]$ and $g\colon\ [a,b]\to\mathbb{R}$ nondecreasing on $[a,b]$ and such that

$$\|U(\tau,t) - U(\tau,s)\|_X \leqslant |f(\tau)|\,|g(t)-g(s)| \quad \text{for all } t,s,\tau\in[a,b],$$

then

$$\left\|\int_0^t \mathrm{D}U(\tau,\sigma)\right\|_X \leqslant \int_0^t |f(s)|\,\mathrm{d}g(s) \quad \text{for all } t\in[a,b].$$

Now, we will recall the concept of a solution to the generalized ODE

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x,t). \tag{2.1}$$

Definition 2.4. Let $\Omega \subset X$ be open and let $F\colon \Omega \times [a,b] \to X$. Then the function $x\colon [a,b] \to X$ is said to be a *solution* of the *generalized ODE* (2.1) on $[a,b]$ whenever

$$x(t) \in \Omega \quad \text{and} \quad x(t) = x(a) + \int_a^t \mathrm{D}F(x(\tau),\sigma) \quad \text{for all } t \in [a,b].$$

A proper class of right-hand sides of equation (2.1) is given by the following definition.

Definition 2.5. Let $h\colon [a,b] \to \mathbb{R}$ be nondecreasing on $[a,b]$, let $\omega\colon [0,\infty) \to \mathbb{R}$ be increasing and continuous on $[0,\infty)$ with $\omega(0) = 0$ and let $\Omega \subset X$ be open. Then $\mathcal{F}(\Omega\times[a,b],h,\omega;X)$ is the set of all functions $F\colon \Omega\times[a,b] \to X$ fulfilling the relations

$$\|F(x,t_2) - F(x,t_1)\|_X \leqslant |h(t_2) - h(t_1)| \tag{2.2}$$

and

$$\|F(x,t_2) - F(x,t_1) - F(y,t_2) + F(y,t_1)\|_X \leqslant \omega(\|x-y\|_X)|h(t_2)-h(t_1)| \tag{2.3}$$
$$\text{for all } x,y \in \Omega \text{ and } t_1,t_2 \in [a,b].$$

If $X = \mathbb{R}^n$, we write $\mathcal{F}(\Omega \times [a,b],h,\omega)$ instead of $\mathcal{F}(\Omega \times [a,b],h,\omega;\mathbb{R}^n)$.

Next two assertions are taken from [4], cf. Lemmas 4.5 and 4.6 therein.

Lemma 2.6. *Assume that $F\colon \Omega \times [a,b] \to X$ fulfils (2.2). Then for any $x \in G([a,b];X)$ such that $x(t) \in \Omega$ for all $t \in [a,b]$, the integral $\int_a^b \mathrm{D}F(x(\tau),\sigma)$ exists and the inequality*

$$\left\|\int_{t_1}^{t_2} \mathrm{D}F(x(\tau),\sigma)\right\|_X \leqslant |h(t_2) - h(t_1)|$$

is true for all $t_1,t_2 \in [a,b]$. Furthermore, the function

$$t \in [a,b] \to \int_a^t \mathrm{D}F(x(\tau),\sigma)$$

has a bounded variation on $[a,b]$.

Finally, every solution x of (2.1) has a bounded variation on $[a,b]$ and, in particular, it is regulated on $[a,b]$.

Lemma 2.7. *Let $F \in \mathcal{F}(\Omega \times [a,b], h, \omega; X)$, where $h\colon [a,b] \to \mathbb{R}$ is nondecreasing on $[a,b]$, $\omega\colon [0,\infty) \to \mathbb{R}$ is increasing and continuous on $[0,\infty)$, $\omega(0) = 0$ and $\Omega \subset X$ is open. Then*

$$\biggl\| \int_{t_1}^{t_2} \mathrm{D}[F(x(\tau),\sigma) - F(y(\tau),\sigma)] \biggr\|_X \leqslant \int_{t_1}^{t_2} \omega(\|x(s) - y(s)\|_X)\, \mathrm{d}h(s)$$

for all $[t_1,t_2] \subset [a,b]$ and $x, y \in G([a,b]; X)$ such that $x(t) \in \Omega$ and $y(t) \in \Omega$ for all $t \in [a,b]$.

Remark 2.8. If we consider in (2.1) a particular case $F(x,t) = A(t)x$, where $A\colon [a,b] \to \mathcal{L}(X)$, we obtain the generalized linear ODE

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}[A(t)x]. \tag{2.4}$$

Obviously, the function $x\colon [a,b] \to X$ is a solution of the generalized linear ODE (2.4) on $[a,b]$ whenever

$$x(t) = x(a) + \int_a^t \mathrm{d}[A(s)]\, x(s) \quad \text{for } t \in [a,b], \tag{2.5}$$

where the integral stands for the Kurzweil-Stieltjes one.

Finally, we state the following basic result from Theorem 4.2 in [43] well illustrating the importance of the class $\mathcal{F}(\Omega \times [a,b], h, \omega)$ in the theory of generalized ODEs. For the analogue in the general case, see Theorem 5.1 in [4].

Theorem 2.9. *Assume there are $h\colon [a,b] \to \mathbb{R}$ nondecreasing on $[a,b]$ and left continuous on $(a,b]$ and $\omega\colon [0,\infty) \to \mathbb{R}$ increasing and continuous on $[0,\infty)$ with $\omega(0) = 0$ such that $F \in \mathcal{F}(\Omega\times [0,T], h, \omega)$. Furthermore, let $(x_0,t_0) \in \Omega \times [a,b)$ be such that $x_0 + F(x_0, t_0+) - F(x_0,t_0) \in \Omega$. Then there is a $\Delta > 0$ such that the equation (2.1) has a solution x on $[t_0, t_0 + \Delta]$ such that $x(t_0) = x_0$.*

3. Bifurcation theory for generalized ODEs

In this section, we will consider the concept of a bifurcation point with respect to a given solution of the parameterized periodic boundary value problem

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(\lambda, x, t), \quad x(0) = x(T). \tag{3.1}$$

In the rest of the paper we have $a = 0$ and $0 < b = T < \infty$. Furthermore, given a Banach space X, the symbol Id stands for identity operator on X and for a given $x_0 \in X$ and $\varrho > 0$ we denote by $B(x_0,\varrho)$ the open ball in X centered at x_0 and with the radius ϱ, while $\overline{B(x_0,\varrho)}$ is its closure.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$ be open and $F\colon \Lambda \times \Omega \times [0,T] \to \mathbb{R}^n$. Then the couple $(x,\lambda) \in G([0,T];\mathbb{R}^n) \times \Lambda$ is a *solution* of the problem (3.1) whenever $x(0) = x(T)$ and

$$x(t) \in \Omega \quad \text{and} \quad x(t) = x(0) + \int_0^t \mathrm{D}F(\lambda, x(\tau), \sigma) \quad \text{for all } t \in [0,T].$$

For our purposes, the following hypotheses will be helpful.

$$\text{(3.2)} \quad \begin{cases} \Omega \subset \mathbb{R}^n \text{ and } \Lambda \subset \mathbb{R} \text{ are open sets; } F\colon \Lambda \times \Omega \times [0,T] \to \mathbb{R}^n \text{ and} \\ \text{there are } h : [0,T] \to \mathbb{R} \text{ nondecreasing and } \omega\colon [0,\infty) \to [0,\infty) \\ \text{increasing and continuous and such that } \omega(0) = 0 \text{ and} \\ F(\lambda,\cdot,\cdot) \in \mathcal{F}(\Omega \times [0,T], h, \omega) \text{ for each } \lambda \in \Lambda; \end{cases}$$

$$\text{(3.3)} \quad \begin{cases} (x_0,\lambda) \in G([0,T];\mathbb{R}^n) \times \Lambda \text{ is a solution of (3.1) for each } \lambda \in \Lambda \text{ and} \\ \text{there is } \varrho > 0 \text{ such that } x(t) \in \Omega \text{ for all } (t,x) \in [0,T] \times \overline{B(x_0,\varrho)}. \end{cases}$$

Furthermore, as in Definition 5.1 of [12], let us define

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t \mathrm{D}F(\lambda, x(\tau), \sigma), \tag{3.4}$$

whenever the Kurzweil integral on the right-hand side has sense.

Proposition 3.2. *Assume* (3.2) *and* (3.3) *and let the operator* Φ *be defined by* (3.4). *Then* $\Phi(\lambda,\cdot)$ *maps* $\overline{B(x_0,\varrho)}$ *into* $G([0,T];\mathbb{R}^n)$ *for any* $\lambda \in \Lambda$. *Moreover, problem* (3.1) *is equivalent to finding solutions* (x,λ) *of the operator equation*

$$x = \Phi(\lambda, x). \tag{3.5}$$

P r o o f. The first part of the statement follows from Lemma 2.7. Furthermore, if

$$x(t) = x(T) + \int_0^t \mathrm{D}F(\lambda, x(\tau), \sigma) \quad \text{for } t \in [0,T], \tag{3.6}$$

then for $t = 0$ we get $x(0) = x(T)$. As a result, (x,λ) is a solution to (3.1). The opposite implication is obvious. □

Furthermore, by [12], Propostion 5.2, for each $\lambda \in \Lambda$, the operator $\Phi(\lambda,\cdot)\colon \overline{B(x_0,\varrho)} \to G([0,T];\mathbb{R}^n)$ is continuous and relatively compact on $\overline{B(x_0,\varrho)}$.

Let us recall also that in Section 4 of [12], classical conditions on the existence of a periodic solution of non autonomous ordinary differential equations are extended to problem (3.6).

We define the notion of the bifurcation point like Krasnoselskiĭ and Zabreĭko did in [21], Section 56:

Definition 3.3. Solution $(x_0, \lambda_0) \in G([0,T]; \mathbb{R}^n) \times \Lambda$ of (3.5) is said to be a *bifurcation point* of (3.5) (i.e., of (3.1)) if every neighborhood of (x_0, λ_0) in $B(x_0, \varrho) \times \Lambda$ contains a solution (x, λ) of (3.5) such that $x \neq x_0$.

Remark 3.4. Note that $(x_0, \lambda_0) \in G([0,T]; \mathbb{R}^n) \times \Lambda$ is a bifurcation point of (3.5) if and only if there is a sequence $\{x_n, \lambda_n\}$ of solutions to (3.5) such that $x_n \to x_0$, $\lambda_n \to \lambda_0$ and $x_n \neq x_0$ for all $n \in \mathbb{N}$.

As usual (cf. e.g. [11], Section 5.2), for a Banach space X, open bounded set $\Omega \subset X$, a compact operator $\Phi\colon \overline{\Omega} \to X$ and $z \notin (\mathrm{Id} - \Phi)(\partial\,\Omega)$, the symbol $\deg_{\mathrm{LS}}(\mathrm{Id} - \Phi, \Omega, z)$ stands for the *Leray-Schauder degree* of $\mathrm{Id} - \Phi$ with respect to Ω, at the point z. Furthermore, if a is an isolated fixed point of Φ, then the value $\mathrm{ind}_{\mathrm{LS}}(\mathrm{Id} - \Phi, a)$ defined by

$$\mathrm{ind}_{\mathrm{LS}}(\mathrm{Id} - \Phi, a) = \deg_{\mathrm{LS}}[\mathrm{Id} - \Phi, B(a,r), 0] \quad \text{for small } r > 0$$

is said to be the *Leray-Schauder index* of $\mathrm{Id} - \Phi$ at a, or sometimes also the *index of an isolated fixed point* of Φ.

Now, by an obvious modification of the proof of Theorem 5.6 in [12] providing conditions sufficient for the existence of the bifurcation point (3.5), we can state its slightly reformulated version.

Theorem 3.5. *Assume* (3.2), (3.3) *and*

$$\text{(3.7)} \qquad \begin{cases} \text{there is a function } \gamma\colon\ [0,T] \to \mathbb{R} \text{ nondecreasing and such that} \\ \text{for any } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ \|F(\lambda_1, x, t) - F(\lambda_2, x, t) - F(\lambda_1, x, s) + F(\lambda_2, x, s)\|_n < \varepsilon|\gamma(t) - \gamma(s)| \\ \text{for } x \in \Omega,\ t, s \in [0,T] \text{ and } \lambda_1, \lambda_2 \in \Lambda \text{ such that } |\lambda_1 - \lambda_2| < \delta. \end{cases}$$

Moreover, let the operator Φ *be defined by* (3.4) *and let* $[\lambda_1^*, \lambda_2^*] \subset \Lambda$ *be such that*

$$\text{(3.8)} \qquad x_0 \text{ is an isolated fixed point of the operators } \Phi(\lambda_1^*, \cdot) \text{ and } \Phi(\lambda_2^*, \cdot)$$

and

$$\text{(3.9)} \qquad \mathrm{ind}_{\mathrm{LS}}(\mathrm{Id} - \Phi(\lambda_1^*, \cdot), x_0) \neq \mathrm{ind}_{\mathrm{LS}}(\mathrm{Id} - \Phi(\lambda_2^*, \cdot), x_0).$$

Then there is $\lambda_0 \in [\lambda_1^*, \lambda_2^*]$ *such that* (x_0, λ_0) *is a bifurcation point of* (3.1).

Our wish is to deliver also conditions which are necessary for the existence of a bifurcation point of equation (3.5). This will be given by Theorem 3.13. Before formulating and proving this theorem let us turn our attention to the following immediate observation:

If (x_0, λ_0) is a solution to (3.5), then by Definition 3.3 it is not a bifurcation point of (3.5) whenever it has a neighborhood $\mathcal{U} \subset B(x_0, \varrho) \times \Lambda$ in $G([0,T];\mathbb{R}^n) \times \mathbb{R}$ such that $x = x_0$ holds for any solution (x, λ) to (3.5) belonging to $\mathcal{U}$. It follows that the set of couples $(x, \lambda) \in G([0,T];\mathbb{R}^n) \times \Lambda$ which are not bifurcation points of (3.5) is open in $G([0,T];\mathbb{R}^n) \times \mathbb{R}$. In particular, we have:

Corollary 3.6. *If (x_0, λ_0) is not a bifurcation point of (3.5), then there is a $\delta > 0$ such that the set $B((x_0, \lambda_0), \delta)$ does not contain any bifurcation point of (3.5).*

Furthermore, in the proof of Theorem 3.13 the notion of the derivative of the operator function Φ is needed.

Definition 3.7. Let X, Y be Banach spaces, $D \subset X$ open and G an operator function mapping D into Y. By the derivative $G'(x)$ of G at the point $x \in D$ we understand its Frechet derivative at x, i.e., $G'(x)$ is the linear bounded operator on X such that

$$\lim_{\vartheta \to 0+} \left\| \frac{G(x + \vartheta\, z) - G(x)}{\vartheta} - G'(x)z \right\|_Y = 0 \quad \text{for all } z \in X.$$

In particular, derivative of $\Phi(\lambda, \cdot)$ at x will be denoted by $\Phi'_x(\lambda, x)$ and similarly, derivative of the function $F(\lambda, \cdot, t)\colon \Omega \to \mathbb{R}^n$ at $x_0 \in \Omega$ is denoted as $F'_x(\lambda, x_0, t)$. Recall that $F'_x(\lambda, x, t) \in \mathcal{L}(\mathbb{R}^n)$ is represented by an $n \times n$-matrix.

Next assertion provides the explicit form of the derivative of the operator $\Phi(\lambda, \cdot)$ given by (3.4).

Proposition 3.8. *Assume that conditions (3.2) and (3.3) are satisfied, Φ is defined by (3.4) and $\varrho > 0$ is given by (3.3). Furthermore, suppose that for each $(\lambda, x, t) \in \Lambda \times \Omega \times [0,T]$ the function F has a derivative $F'_x(\lambda, x, t)$ which is for each $(\lambda, t) \in \Lambda \times [0,T]$ continuous with respect to x on Ω and such that*

$$(3.10) \quad \begin{cases} F'_x(\lambda, \cdot, \cdot) \in \mathcal{F}(\Omega \times [0,T], \widetilde{h}, \widetilde{\omega}; \mathcal{L}(\mathbb{R}^n)) \text{ for all } \lambda \in \Lambda, \text{ where} \\ \widetilde{h}\colon [0,T] \to [0,\infty) \text{ is nondecreasing on } [a,b] \quad \text{and} \\ \widetilde{\omega}\colon [0,\infty) \to [0,\infty) \text{ is continuous and increasing on } [0,\infty) \text{ and } \widetilde{\omega}(0) = 0. \end{cases}$$

Then for each $(\lambda, x) \in \Lambda \times B(x_0, \varrho)$ the derivative $\Phi'_x(\lambda, x)$ of $\Phi(\lambda, \cdot)$ at x is given by

$$(3.11) \qquad (\Phi'_x(\lambda, x)\, z)(t) = z(T) + \int_0^t \mathrm{D}[F'_x(\lambda, x(\tau), \sigma)\, z(\tau)] \qquad \text{for } z \in G([0,T];\mathbb{R}^n) \text{ and } t \in [0,T].$$

P r o o f. First, recall that $F'_x(\lambda,\cdot,\cdot)\in\mathcal{F}(\Omega\times[0,T],\widetilde{h},\widetilde{\omega};\mathcal{L}(\mathbb{R}^n))$ means that

$$\|F'_x(\lambda,x,t)-F'_x(\lambda,x,s)\|_{n\times n}\leqslant|\widetilde{h}(t)-\widetilde{h}(s)| \text{ for } \lambda\in\Lambda,\ x\in\Omega \text{ and } t,s\in[0,T]. \tag{3.12}$$

and

$$\begin{aligned}\|F'_x(\lambda,x,t)-F'_x(\lambda,x,s)-F'_x(\lambda,y,t)+F'_x(\lambda,y,s)\|_{n\times n}&\\ \leqslant\widetilde{\omega}(\|x-y\|_n)\,|\widetilde{h}(t)-\widetilde{h}(s)| \quad \text{for } \lambda\in\Lambda,\ x,y\in\Omega \text{ and } t,s\in[0,T].&\end{aligned}\tag{3.13}$$

By Proposition 3.2, Φ maps $\overline{B(x_0,\varrho)}$ into $G([0,T];\mathbb{R}^n)$ for any $\lambda\in\Lambda$. Let $x\in B(x_0,\varrho)$ and $\lambda\in\Lambda$ be given. By (3.3), $x(t)\in\Omega$ for all $t\in[0,T]$. Consider the operator function Ψ defined by

$$(\Psi(\lambda,x)\,z)(t)=z(T)+\int_0^t \mathrm{D}[F'_x(\lambda,x(\tau),\sigma)\,z(\tau)] \quad \text{for } z\in G([0,T];\mathbb{R}^n) \text{ and } r\in[0,T].$$

Obviously, $\Psi(\lambda,x)\colon G([0,T];\mathbb{R}^n)\to G([0,T];\mathbb{R}^n)$ is linear and bounded. Indeed, by Lemma 2.3 and (3.12) we have

$$\begin{aligned}\|\Psi(\lambda,x)\,z\|_\infty&=\sup_{t\in[0,T]}\|(\Psi(\lambda,x)\,z)(t)\|_n=\sup_{t\in[0,T]}\left\|z(T)+\int_0^t \mathrm{D}[F'_x(\lambda,x(\tau),\sigma)z(\tau)]\right\|_n\\ &\leqslant\|z(T)\|_n+\sup_{t\in[0,T]}\int_0^t\|z(\tau)\|_n\,\mathrm{d}\widetilde{h}(\tau)\leqslant[1+(\widetilde{h}(T)-\widetilde{h}(0))]\|z\|_\infty\end{aligned}$$

for each $z\in G([0,T];\mathbb{R}^n)$.

We want to show that

$$\lim_{\vartheta\to0+}\left\|\frac{\Phi(\lambda,x+\vartheta\,z)-\Phi(\lambda,x)}{\vartheta}-\Psi(\lambda,x)\,z\right\|_\infty=0 \quad \text{for all } z\in G([0,T];\mathbb{R}^n). \tag{3.14}$$

To this aim, let $z\in G([0,T];\mathbb{R}^n)$ be given. Then for every $t\in[0,T]$ and $\vartheta\in(0,1)$ sufficiently small we have $x+\vartheta\,z\in B(x_0,\varrho)$ and

$$\frac{\Phi(\lambda,x+\vartheta\,z)(t)-\Phi(\lambda,x)(t)}{\vartheta}-(\Psi(\lambda,x)\,z)(t)=\int_0^t \mathrm{D}U(\tau,\sigma),$$

where

$$\begin{aligned}U(\tau,\sigma)=&\frac{F(\lambda,x(\tau)+\vartheta\,z(\tau),\sigma)-F(\lambda,x(\tau),\sigma)}{\vartheta}\\ &-F'_x(\lambda,x(\tau),\sigma)\,z(\tau) \quad \text{for } \tau,\sigma\in[0,T].\end{aligned}\tag{3.15}$$

Notice, that due to the convexity of $B(x_0, \varrho)$, the functions $\alpha\,(x + \vartheta\, z) + (1-\alpha)\, x$ belong to $B(x_0, \varrho)$ for each $\alpha \in [0,1]$. In particular, $\alpha\,(x(\tau){+}\vartheta\, z(\tau)){+}(1{-}\alpha)\, x(\tau) \in \Omega$ for all $\tau \in [0,T]$ and $\alpha \in [0,1]$. Thus, we can use the Mean Value Theorem for vector-valued functions (see e.g. [20], Lemma 8.11) to verify that the relations

$$\begin{aligned} &F(\lambda, x(\tau) + \vartheta z(\tau), \sigma) - F(\lambda, x(\tau), \sigma) \\ &\qquad = \left[\int_0^1 F'_x(\lambda, \alpha(x(\tau) + \vartheta z(\tau)) + (1-\alpha)x(\tau), \sigma)\,\mathrm{d}\alpha\right]\vartheta z(\tau) \end{aligned}$$

are true for arbitrary $\tau, \sigma \in [0,T]$. Therefore, we can rearrange the difference $U(\tau,t) - U(\tau,s)$ as follows:

$$\begin{aligned} U(\tau,t) - U(\tau,s) = \Bigg[\int_0^1 &[F'_x(\lambda, \alpha(x(\tau){+}\vartheta z(\tau)) + (1-\alpha)x(\tau), t) \\ &- F'_x(\lambda, \alpha(x(\tau) + \vartheta\, z(\tau)) + (1-\alpha)x(\tau), s)]\,\mathrm{d}\alpha \\ &- \int_0^1 [F'_x(\lambda, x(\tau), t) - F'_x(\lambda, x(\tau), s)]\,\mathrm{d}\alpha\Bigg] z(\tau) \\ &\qquad \text{for}\, t, s, \tau \in [0,T]. \end{aligned} \tag{3.16}$$

Furthermore, using (3.13) we obtain

$$\begin{aligned} &\|F'_x(\lambda, \alpha(x(\tau) + \vartheta\, z(\tau)) + (1{-}\alpha)x(\tau), t) - F'_x(\lambda, \alpha\,(x(\tau) \\ &\qquad + \vartheta\, z(\tau)) + (1{-}\alpha)x(\tau), s) - F'_x(\lambda, x(\tau), t) + F'_x(\lambda, x(\tau), s)\|_{n\times n} \\ &\qquad \leqslant \widetilde{\omega}(\vartheta\|z\|_\infty)|\widetilde{h}(t) - \widetilde{h}(s)| \quad \text{for } \vartheta \in [0,1] \text{ and } t, s, \tau \in [0,T]. \end{aligned} \tag{3.17}$$

Inserting (3.17) into (3.16), we verify that the inequality

$$\|U(\tau,t) - U(\tau,s)\|_n \leqslant \widetilde{\omega}(\vartheta\|z\|_\infty)|\widetilde{h}(t) - \widetilde{h}(s)|\|z\|_\infty$$

holds for all $t, s, \tau \in [0,t]$. Finally, making use of Lemma 2.3 we achieve the inequality

$$\sup_{t\in[0,T]}\left\|\int_0^t \mathrm{D}U(\tau,\sigma)\right\|_n \leqslant \int_0^T \widetilde{\omega}(\vartheta\|z\|_\infty)\,\mathrm{d}\widetilde{h}\|z\|_\infty = \widetilde{\omega}(\vartheta\|z\|_\infty)[\widetilde{h}(T) - \widetilde{h}(0)]\|z\|_\infty.$$

This together with (3.15) implies the relations

$$\begin{aligned} 0 &\leqslant \lim_{\vartheta\to 0+}\left\|\frac{\Phi(\lambda, x + \vartheta\, z) - \Phi(\lambda, x)}{\vartheta} - \Psi(\lambda, x)z\right\|_\infty \\ &\leqslant \lim_{\vartheta\to 0+} \widetilde{\omega}(\vartheta\|z\|_\infty)[\widetilde{h}(T) - \widetilde{h}(0)]\|z\|_\infty = 0, \end{aligned}$$

i.e., the desired relation (3.14) is true. This completes the proof. □

Next proposition shows that when we include, in addition, conditions (3.7), we reach the continuity of Φ on $\Lambda \times B(x_0, \varrho)$.

Proposition 3.9. *Assume that* (3.2), (3.3), (3.7) *are satisfied and let* Φ *be given by* (3.4). *Then* Φ *is continuous on* $\Lambda \times \overline{B(x_0, \varrho)}$.

P r o o f. Let $(\lambda_1, x), (\lambda_2, y) \in \Lambda \times \overline{B(x_0, \varrho)}$ and $t \in [0, T]$ be given. Obviously, we have

$$(3.18)\quad [\Phi(\lambda_1, x) - \Phi(\lambda_2, y)](t) = x(T) - y(T) + \int_0^t \mathrm{D}[F(\lambda_1, x(\tau), \sigma) - F(\lambda_2, y(\tau), \sigma)],$$

where

$$\begin{aligned}\int_0^t & \mathrm{D}[F(\lambda_1, x(\tau), \sigma) - F(\lambda_2, y(\tau), \sigma)] \\ &= \int_0^t \mathrm{D}[F(\lambda_1, x(\tau), \sigma) - F(\lambda_1, y(\tau), \sigma)] + \int_0^t \mathrm{D}[F(\lambda_1, y(\tau), \sigma) - F(\lambda_2, y(\tau), \sigma)].\end{aligned}$$

Furthermore,

$$(3.19)\qquad \Big\| \int_0^t \mathrm{D}[F(\lambda_1, x(\tau), \sigma) - F(\lambda_1, y(\tau), \sigma)] \Big\|_n \leqslant \omega(\|x - y\|_\infty)[h(T) - h(0)]$$

due to Lemma 2.6.

Now, let $\varepsilon > 0$ be given and let $\delta \in (0, \varepsilon)$ be such that (3.7) is true. Then Lemma 2.3 implies that also the relation

$$\Big\| \int_0^t \mathrm{D}[F(\lambda_1, y(\tau), \sigma) - F(\lambda_2, y(\tau), \sigma)] \Big\|_n < \varepsilon\, [\gamma(T) - \gamma(0)]$$

holds whenever $|\lambda_1 - \lambda_2| < \delta$. To summarize, inserting the last relation together with (3.19) into (3.18) we obtain

$$\begin{aligned}\|\Phi(\lambda_1, x) - \Phi(\lambda_2, y)\|_\infty &\leqslant \|x - y\|_\infty + \omega(\|x - y\|_\infty)\, [h(T) - h(0)] + \varepsilon[\gamma(T) - \gamma(0)] \\ &< \varepsilon(1 + [h(T) - h(0)] + [\gamma(T) - \gamma(0)])\end{aligned}$$

whenever $\|x - y\|_\infty$ is sufficiently small. In other words, the operator function Φ is continuous on $\Lambda \times \overline{B(x_0, \varrho)}$. □

Similarly, by adding a requirement analogous to (3.7), but with the derivative F'_x in the place of F, we achieve the continuity of the derivative Φ'_x on $\Lambda \times B(x_0, \varrho)$.

Proposition 3.10. *Let the assumptions of Proposition* 3.8 *be satisfied and let*

$$(3.20)\qquad \begin{cases} \text{there be a nondecreasing function } \widetilde{\gamma}\colon\ [0,T] \to \mathbb{R} \text{ such that for} \\ \text{any } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that} \\ \|F'_x(\lambda_1,x,t) - F'_x(\lambda_2,x,t) - F'_x(\lambda_1,x,s) + F'_x(\lambda_2,x,s)\|_{n\times n} \\ < \varepsilon|\widetilde{\gamma}(t) - \widetilde{\gamma}(s)| \text{ for } x \in \Omega,\ t,s \in\ [0,T] \text{ and } \lambda_1,\lambda_2 \in \Lambda \\ \text{such that } |\lambda_1 - \lambda_2| < \delta. \end{cases}$$

Then the operator function $\Phi'_x\colon\ \Lambda \times B(x_0,\varrho) \to \mathcal{L}(G(\,[0,T];\mathbb{R}^n))$ *is continuous.*

P r o o f. The proof is quite analogous to that of Proposition 3.9, only instead of $\Phi(\lambda,x)$ and $F(\lambda,x(\tau),t)$ we should, respectively, deal with $\Phi'_x(\lambda,x)\,z$ and $F'_x(\lambda,x(\tau),t)z(\tau)$, where $z \in G(\,[0,T];\mathbb{R}^n)$. □

Now, we are prepared to formulate and prove one of the main new results of this section.

Theorem 3.11. *Let* (3.7) *and all the assumptions of Proposition* 3.10 *be satisfied, let* $\lambda_0 \in \Lambda$ *be given and let* $\mathrm{Id} - \Phi'_x(\lambda_0,x_0)$ *be an isomorphism of* $G(\,[0,T];\mathbb{R}^n)$ *onto* $G(\,[0,T];\mathbb{R}^n)$. *Then there is* $\delta > 0$ *such that* (x,λ) *is not a bifurcation point of the equation* $\Phi(\lambda,x) = x$ *whenever* $\|x - x_0\|_\infty + |\lambda - \lambda_0| < \delta$.

P r o o f. First, recall that according to Propositions 3.8, 3.9 and 3.10, the operator function $\Phi(\lambda,\cdot)$ is continuous together with its derivative $\Phi'_x(\lambda,x) \in \mathcal{L}(G(\,[0,T];\mathbb{R}^n))$ on $\Lambda \times B(x_0,\varrho)$. Further, by (3.3) we have

$$(3.21)\qquad x_0 = \Phi(\lambda,x_0) \quad \text{for all } \lambda \in \Lambda.$$

Let $\mathrm{Id} - \Phi'_x(\lambda_0,x_0)$ be an isomorphism of $G(\,[0,T];\mathbb{R}^n)$ onto $G(\,[0,T];\mathbb{R}^n)$. By the Implicit Function Theorem (see e.g. [11], Theorem 4.2.1) this means that there exist neighborhoods $\mathcal{V} \subset \Lambda$ of λ_0 and $\mathcal{W} \subset B(x_0,\varrho)$ of x_0 such that for any $\lambda \in \mathcal{V}$ there is a unique $x \in \mathcal{W}$ such that $x = \Phi(\lambda,x)$. However, this together with (3.21) implies that $x = x_0$ has to be the only function satisfying the relations

$$x = \Phi(\lambda,x) \quad \text{for any } \lambda \in \mathcal{V} \subset \Lambda.$$

Hence, according to Definition 3.3, (x_0,λ_0) is not a bifurcation point of the equation $x = \Phi(\lambda,x)$. The proof will be completed by using Corollary 3.6. □

Next assertion provides a useful Fredholm Alternative type result.

Proposition 3.12. *Let the assumptions of Proposition* 3.8 *be satisfied and let* $x_0 \in G(\,[0,T];\mathbb{R}^n)$ *be given. Then either*

(i) *the equation*

$$z(t) - z(T) - \int_0^t \mathrm{D}[F'_x(\lambda_0, x_0(\tau), \sigma) z(\tau)] = q(t) \quad \textit{for} \;\; t \in [0,T]$$

has a unique solution in $G([0,T];\mathbb{R}^n)$ *for every* $q \in G([0,T];\mathbb{R}^n)$; *or*

(ii) *the corresponding homogeneous equation*

$$z(t) - z(T) - \int_0^t \mathrm{D}[F'_x(\lambda_0, x_0(\tau), \sigma) z(\tau)] = 0 \quad \textit{for}\; t \in [0,T]$$

has at least one nontrivial solution in $G([0,T];\mathbb{R}^n)$.

Proof. Let Φ be defined by (3.4) and let $\varrho > 0$ be given by (3.3). Let $\lambda \in \Lambda$ and $x \in B(x_0, \varrho)$ be given. By Proposition 3.8, we have

$$(\Phi'_x(\lambda, x)\, z)(t) = z(T) + \int_0^t \mathrm{D}[F'_x(\lambda, x(\tau), \sigma) z(\tau)] \quad \text{for } z \in G([0,T];\mathbb{R}^n) \text{ and } t \in [0,T].$$

We assert that $\Phi'_x(\lambda, x)$ is a linear compact operator on $G([0,T];\mathbb{R}^n)$. Indeed, it is linear and bounded as it was shown at the beginning of the proof of Proposition 3.8. Hence, it remains to show that it maps bounded subsets of $G([0,T];\mathbb{R}^n)$ onto relatively compact subsets of $G([0,T];\mathbb{R}^n)$.

Let $M \subset G([0,T];\mathbb{R}^n)$ be bounded and let $c > 0$ be such that $\|z\|_\infty \leqslant c$ for all $z \in M$. Making use of (3.12) and Lemma 2.3, we get

$$\begin{aligned} \|(\Phi'_x(\lambda, x) z)(t') - (\Phi'_x(\lambda, x) z)(t)\|_\infty &= \left\| \int_t^{t'} \mathrm{D}[F'_x(\lambda, x(\tau), \sigma) z(\tau)] \right\|_n \\ &\leqslant \int_{\min\{t,t'\}}^{\max\{t,t'\}} \|z(\tau)\|_n \,\mathrm{d}\widetilde{h}(\tau) \leqslant c|\widetilde{h}(t') - \widetilde{h}(t)| \end{aligned}$$

for all $t, t' \in [0,T]$ and $z \in M$. By [13], Theorem 2.17 (cf. also [32], Corollary 4.3.8), the set $\{\Phi'_x(\lambda, x) z\colon\; z \in M\}$ is relatively compact. This proves our claim.

Therefore, using the Fredholm Alternative for Banach spaces (see e.g. [41], Theorem 4.12), we have that either the range $\mathcal{R}(\mathrm{Id} - \Phi'_x(\lambda, x))$ of the operator $\mathrm{Id} - \Phi'_x(\lambda, x)$ is the whole $G([0,T];\mathbb{R}^n)$ and its null space $\mathcal{N}(\mathrm{Id} - \Phi'_x(\lambda, x)) = \{0\}$ or $\mathcal{R}(\mathrm{Id} - \Phi'_x(\lambda, x)) \neq G([0,T];\mathbb{R}^n)$ and $\mathcal{N}(\mathrm{Id} - \Phi'_x(\lambda, x)) \neq \{0\}$. This completes the proof. $\square$

Now, we can reformulate conditions necessary for (λ_0, x_0) to be a bifurcation point of the equation $\Phi(\lambda, x) = x$ as follows:

Theorem 3.13. *Suppose that the assumptions of Theorem* 3.11 *are satisfied and let* $\lambda_0 \in \Lambda$ *and* $x_0 \in B(x_0, \varrho)$ *be given. Then* (x_0, λ_0) *is a bifurcation point of the equation* $\Phi(\lambda, x) = x$ *only if there exists* $q \in G([0,T];\mathbb{R}^n)$ *such that the equation*

$$z(t) - z(T) - \int_0^t \mathrm{D}[F'_x(\lambda_0, x_0(\tau), \sigma) z(\tau)] = q(t) \quad \textit{for } t \in [0,T] \tag{3.22}$$

has no solution in $G([0,T];\mathbb{R}^n)$ *and the corresponding homogeneous equation*

$$z(t) - z(T) - \int_0^t \mathrm{D}[F'_x(\lambda_0, x_0(\tau), \sigma) z(\tau)] = 0 \quad \textit{for } t \in [0,T]$$

possesses at least one nontrivial solution in $G([0,T];\mathbb{R}^n)$.

P r o o f. Suppose (x_0, λ_0) is a bifurcation point of the equation $\Phi(\lambda, x) = x$. Then by Theorem 3.11, the operator $\mathrm{Id} - \Phi'_x(\lambda_0, x_0)\colon G([0,T];\mathbb{R}^n) \to G([0,T];\mathbb{R}^n)$ cannot be an isomorphism. Therefore, using Theorem 3.11 and Fredholm type Alternative 3.12, we conclude that

$$\mathcal{R}(\mathrm{Id} - \Phi'_x(\lambda_0, x_0)) \neq G([0,T];\mathbb{R}^n) \quad \text{and} \quad \mathcal{N}(\mathrm{Id} - \Phi'_x(\lambda_0, x_0)) \neq \{0\}.$$

Our statement follows immediately. □

Remark 3.14. Notice that (3.22) is the periodic problem for a nonhomogeneous generalized linear differential equation.

4. Measure differential equations

Main topic of this paper are measure differential equations of the form

$$\mathrm{D}x = f(\lambda, x, t) + g(x, t) \cdot \mathrm{D}u, \tag{4.1}$$

where

$$\begin{cases} \Omega \subset \mathbb{R}^n \text{ and } \Lambda \subset \mathbb{R} \text{ are open sets, } x\colon [0,T] \to \mathbb{R}^n; \\ f\colon \Lambda \times \Omega \times [0,T] \to \mathbb{R}^n, \ g\colon \Omega \times [0,T] \to \mathbb{R}^n; \\ u\colon (-\infty, T] \to \mathbb{R} \text{ is left-continuous and has a bounded} \\ \text{variation on } [0,T] \text{ and } u(t) = u(0) \text{ for } t < 0; \\ \mathrm{D}x \text{ is the (Schwartz) distributional derivative of } x; \\ \mathrm{D}u \text{ is the (Schwartz) distributional derivative of } u. \end{cases} \tag{4.2}$$

It is well known that such kind of differential equations, usually called distributional or measure differential equations, encompass many types of equations such as ordinary differential equations, impulsive differential equations, dynamic equations on time scales and others.

Remark 4.1. (Distributions.) By distributions we understand linear continuous functionals on the topological vector space $\mathcal{D}$ of functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ possessing for any $j \in N \cup \{0\}$ a derivative $\varphi^{(j)}$ of the order j which is continuous on $\mathbb{R}$ and such that $\varphi^{(j)}(t) = 0$ if $t \notin (0,T)$. The space $\mathcal{D}$ is endowed with the topology in which the sequence $\varphi_k \in \mathcal{D}$ tends to $\varphi_0 \in \mathcal{D}$ in $\mathcal{D}$ if and only if

$$\lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\|_\infty = 0 \text{ for all nonnegative integers } j.$$

Similarly, n-vector distributions are linear continuous n-vector functionals on the nth cartesian power $\mathcal{D}^n$ of $\mathcal{D}$. The space of n-vector distributions on $[0,T]$ (the dual space to $\mathcal{D}^n$) is denoted by $\mathcal{D}^{n*}$. Instead of $\mathcal{D}^{1*}$ we write $\mathcal{D}^*$. Given a distribution $f \in \mathcal{D}^{n*}$ and a (test) function $\varphi \in \mathcal{D}^n$, the value of the functional f on φ is denoted by $\langle f, \varphi \rangle$. Of course, reasonable real valued point functions are naturally included between distributions. For example, for a given f Lebesgue integrable on $[0,T]$ ($f \in L^1([0,T], \mathbb{R}^n)$), the relation

$$\langle f, \varphi \rangle = \int_0^T f(t)\varphi(t)\,\mathrm{d}t \quad \text{for } \varphi \in \mathcal{D}^n$$

(where $f(t)\varphi(t)$ stands for the scalar product of $f(t) \in \mathbb{R}^n$ and $\varphi(t) \in \mathbb{R}^n$) defines the n-vector distribution on $[0,T]$, which will be denoted by the same symbol f. As a result, the zero distribution $0 \in \mathcal{D}^{n*}$ on $[0,T]$ can be identified with an arbitrary measurable function vanishing a.e. on $[0,T]$. Obviously, if $f \in G([0,T];\mathbb{R}^n)$ is left-continuous on $(0,T]$, then $f = 0 \in \mathcal{D}^{*n}$ if and only if $f(t) = 0$ for all $t \in [0,T]$.

Given two distributions $f, g \in \mathcal{D}^{n*}$, $f = g$ means that $f - g = 0 \in \mathcal{D}^{n*}$. Whenever a relation of the form $f = g$ for distributions and/or functions f and g occurs in the following text, it is understood as the equality in the above sense. Given an arbitrary $f \in \mathcal{D}^{n*}$, the symbol $\mathrm{D}f$ denotes its distributional derivative, i.e.,

$$\langle \mathrm{D}f, \varphi \rangle = -\langle f, \varphi' \rangle \quad \text{for } \varphi \in \mathcal{D}^n.$$

For absolutely continuous functions their distributional derivatives coincide with their classical derivatives, of course. It is well-known, cf. [16], Section 3, that if $f \in \mathcal{D}^*$, then $\mathrm{D}f = 0$ if and only if f is Lebesgue integrable on $[0,T]$ and there is a $c_0 \in \mathbb{R}$ such that $f(t) = c_0$ a.e. on $[0,T]$.

For more details on the theory of distributions, see e.g. [14], [19], [32], [38], [46].

Definition 4.2. By a solution of (4.1) we understand a couple $(x,\lambda) \in G([0,T];\mathbb{R}^n) \times \Lambda$ such that x is left-continuous on $(0,T]$, $x(t) \in \Omega$ for $t \in [0,T]$, the distributional product $\widetilde{g}_x \cdot \mathrm{D}u$ of the function

$$\widetilde{g}_x\colon\ t \in [0,T] \to g(x(t),t) \in \mathbb{R}^n$$

with the distributional derivative $\mathrm{D}u$ of u has a sense and equality (4.1) is satisfied in the distributional sense, i.e.,

$$\langle \mathrm{D}x, \varphi\rangle = \langle \widetilde{f}_{\lambda,x}, \varphi\rangle + \langle \widetilde{g}_x \cdot \mathrm{D}u, \varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}^n,$$

where $\widetilde{f}_{\lambda,x}\colon\ t \in [0,T] \to f(\lambda, x(t), t) \in \mathbb{R}^n$.

Remark 4.3. According to Definition 4.2, to investigate differential equations like (4.1), one should reasonably specify how understand to the distributional product $\widetilde{g}_x{\cdot}\mathrm{D}u$, symbolically written as $g(x,t){\cdot}\mathrm{D}u$, on the right-hand side of equation (4.1). It is known that in the Schwartz setting it is not possible to define a product of an arbitrary couple of distributions. In text-books one can find the trivial example when $f \in \mathcal{D}^*$ and $g \in D$. The product $f \cdot g$ of f and g is in such a case defined as

$$\langle f\,g, \varphi\rangle = \langle f, g\varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}.$$

Furthermore, if $f, g \in L^1[0,T]$ are such that $f\,g \in L^1[0,T]$, their distributional product is defined as

$$\langle fg, \varphi\rangle = \int_0^T f(t)g(t)\varphi(t)\,\mathrm{d}t \quad \text{for } \varphi \in \mathcal{D}^n.$$

Thus, in this case the distributional product actually coincides with the usual product of point functions. However, in equation (4.1) we have a product of an n-vector valued function with the distributional derivative of a scalar function, which is evidently not covered by the above definitions. The definition of a product of measures and regulated functions given by Ligęza in [26] on the basis of the sequential approach is unfortunately not suitable for our purposes. As will be seen below, a good tool in the context of measure differential systems is provided by the Kurzweil-Stieltjes integral. The following definition has been introduced in [46], cf. also [32], Section 8.4.

Definition 4.4. If $g\colon\ [0,T] \to \mathbb{R}^n$ and $u\colon\ [0,T] \to \mathbb{R}$ are functions defined on $[0,T]$ and such that there exists the Kurzweil-Stieltjes integral $\int_0^T g\,\mathrm{d}u$, then the product of g and $\mathrm{D}u$ is the distributional derivative of the indefinite integral $G(t) := \int_0^t g\,\mathrm{d}u$, i.e. $g \cdot \mathrm{D}u = \mathrm{D}G$.

Remark 4.5. Note that in Definition 4.4, the product $g \cdot \mathrm{D}u$ is an n-vector distribution.

Furthermore, it is worth mentioning that the multiplication operation given by Definition 4.4 is associative, distributive and multiplication by zero element gives zero element. On the other hand, we should have in mind that (cf. [32], Theorem 6.4.2 and [46], Remark 4.1) the expected formula

$$\mathrm{D}(f \cdot g) = \mathrm{D}f \cdot g + f \cdot \mathrm{D}g$$

for the differentiation of the product $f.g$ is not true, in general. More precisely, using the modified integration-by-parts formula from [28], Theorem 6.2, one can verify that the following relation holds if f and g are regulated and at least one of them has a bounded variation

$$\mathrm{D}(f\cdot g)=\mathrm{D}f\cdot g+f\cdot\mathrm{D}g+\mathrm{D}f\cdot\Delta^{+}\widetilde{g}-\Delta^{-}\widetilde{f}\cdot\mathrm{D}g,$$

where

$$\Delta^{+}\widetilde{g}(t)=\begin{cases}\Delta^{+}g(t) & \text{if } t<T,\\ 0 & \text{if } t=T\end{cases}\quad\text{and}\quad \Delta^{-}\widetilde{f}(t)=\begin{cases}0 & \text{if } t=0,\\ \Delta^{-}f(t) & \text{if } t>0.\end{cases}$$

Together with (4.1) we will consider the Stieltjes integral equation

$$x(t)=x(0)+\int_0^t f(\lambda,x(s),s)\,\mathrm{d}s+\int_0^t g(x(s),s)\,\mathrm{d}u(s)\quad\text{for } t\in[0,T],\tag{4.3}$$

where the integrals stand for the Kurzweil-Stieltjes ones[1].

By a solution we understand any function $x\colon [0,T]\to\mathbb{R}^n$ such that $x(t)\in\Omega$ for $t\in[0,T]$ and the equality (4.3) is true on $[0,T]$.

Remark 4.6. In the literature one often meets instead of the integral version (4.3) of (4.1) the integral equation

$$x(t)=x(0)+\int_0^t f(\lambda,x(s),s)\,\mathrm{d}s+\int_{[0,t)} g(x(s),s)\,\mathrm{d}\mu_u,\tag{4.4}$$

where the former integral is the Lebesgue one and the latter is the Lebesgue-Stieltjes integral. However, it is known, cf. [32], Theorem 6.12.3, that if the Lebesgue-Stieltjes integral (LS) $\int_{[0,T)} g\,\mathrm{d}\mu_u$ exists, then the Kurzweil-Stieltjes integral $\int_0^T g\,\mathrm{d}u$ exists as well and[2]

$$\int_0^T g\,\mathrm{d}u=(\mathrm{LS})\int_{[0,T)} g\,\mathrm{d}\mu_u.$$

Therefore, equation (4.4) is a special case of (4.3).

Proposition 4.7. *Assume that conditions* (4.2),

$$\begin{cases} f(\lambda,\cdot,t) \text{ is continuous on } \Omega \text{ for all } t\in[0,T] \text{ and } \lambda\in\Lambda;\\ f(\lambda,x,\cdot) \text{ is Lebesgue measurable on } [0,T] \text{ for all } (\lambda,x)\in\Lambda\times\Omega;\\ \text{there is a function } m\colon [0,T]\to[0,\infty) \text{ Lebesgue integrable}\\ \text{on } [0,T] \text{ and such that } \|f(\lambda,x,t)\|_n\leqslant m(t)\\ \text{for } (\lambda,x,t)\in\Lambda\times\Omega\times[0,T]\end{cases}\tag{4.5}$$

[1] Recall that the Kurzweil-Stieltjes integral with the identity integrator becomes the Henstock-Kurzweil one.

[2] Recall that u is left-continuous on $(0,T]$ and $u(0-)=u(0)$.

and

$$(4.6)\qquad \begin{cases} g(\cdot,t) \text{ is continuous on } \Omega \text{ for all } t\in [0,T] \text{ and there is}\\ \text{a function } m_u\colon [0,T]\to[0,\infty) \text{ such that}\\ \|g(x,t)\|_n \leqslant m_u(t) \text{ and } \int_0^T m_u(t)\,\mathrm{d}[\mathrm{var}_0^t u]<\infty\\ \text{for } (x,t)\in\Omega\times [0,T], \end{cases}$$

are satisfied.

Then any solution x of (4.3) *on* $[0,T]$ *is left-continuous on* $(0,T]$ *and has a bounded variation on* $[0,T]$.

P r o o f. Let x be a solution of (4.3). Then $x(t)\in\Omega$ for all $t\in[0,T]$ and both integrals on the right-hand side of (4.3) have a sense for all $t\in[0,T]$. Due to condition (4.5), the integral $\int_0^T f(\lambda,x(s),s)\,\mathrm{d}s$ exists as the Lebesgue one and as a result the corresponding indefinite integral is absolutely continuous on $[0,T]$.

Furthermore, denote

$$G(t):=\int_0^t g(x(s),s)\,\mathrm{d}u(s)\quad \text{for } t\in[0,T].$$

By [32], Corollary 6.5.5, G is left-continuous on $(0,T]$. Furthermore, due to (4.6) and [32], Theorem 6.7.4, the integral $\int_c^d \|g(x(s),s)\|_n\,\mathrm{d}[\mathrm{var}_0^s u]$ exists for each $[c,d]\subset[0,T]$. Consequently, [32], Theorem 6.3.6 yields the inequalities

$$\sum_{j=1}^m \|G(\alpha_j)-G(\alpha_{j-1})\|_n \leqslant \sum_{j=1}^m \int_{\alpha_{j-1}}^{\alpha_j} \|g(x(s),s)\|_n\,\mathrm{d}[\mathrm{var}_0^s u] \leqslant \int_0^T m_u(s)\,\mathrm{d}[\mathrm{var}_0^s u]<\infty$$

for all divisions $\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ of $[0,T]$, i.e., G has a bounded variation on $[0,T]$. The proof now immediately follows. □

Now, we will describe the relationship between measure differential system (4.1) and integral system (4.3).

Theorem 4.8. *Let conditions* (4.2), (4.5) *and* (4.6) *be satisfied. Then* $x\in G([0,T];\mathbb{R}^n)$ *is a solution of* (4.1) *on* $[0,T]$ *if and only if it is a solution to* (4.3).

P r o o f. If x is a solution to (4.3), then it is a solution to (4.1) on $[0,T]$ thanks to Proposition 4.7 and Definition 4.4.

On the other hand, let x be a solution of (4.1). By Definition 4.2, x is left-continuous on $(0,T]$, has a bounded variation on $[0,T]$ and $x(t)\in\Omega$ for all $t\in[0,T]$. Furthermore, by Definition 4.4,

$$D(x-F_\lambda(x))=0\in\mathcal{D}^{n*},$$

where

$$F_\lambda(x)\colon\ t\in[0,T]\to\int_0^t f(\lambda,x(s),s)\,\mathrm{d}s+\int_0^t g(x(s),s)\,\mathrm{d}u(s)\in\mathbb{R}^n\quad\text{for }\lambda\in\Lambda.$$

By the proof of Proposition 4.7, $F_\lambda(x)$ has a bounded variation on $[0,T]$ and is left-continuous on $(0,T]$ for all $\lambda\in\Lambda$. By [16], Section 3 this means that there is $c\in\mathbb{R}^n$ such that $x(t)-F_\lambda(x)(t)=c$ for all $\lambda\in\Lambda$ and $t\in[0,T]$. As a result, $c=x(0)$ and x is a solution to (4.3). □

Let us consider the function F given for $(\lambda,x,t)\in\Lambda\times\Omega\times[0,T]$ by the relation

$$F(\lambda,x,t)=\int_0^t f(\lambda,x,s)\,\mathrm{d}s+\int_0^t g(x,s)\,\mathrm{d}u(s)\tag{4.7}$$

whenever the integrals on the right-hand sides have a sense.

Next two assertions follow immediately from [42], Propositions 4.7 and 4.8, respectively.

Proposition 4.9. *Let the assumptions of Theorem* 4.8 *be satisfied and let* F *be given by* (4.7)*. Then there are a nondecreasing function* $h\colon[0,T]\to\mathbb{R}$ *left-continuous on* $(0,T]$ *and a continuous, increasing function* $\omega\colon[0,\infty)\to\mathbb{R}$ *with* $\omega(0)=0$ *and such that* $F(\lambda,\cdot,\cdot)\in\mathcal{F}(\Omega\times[0,T],h,\omega)$ *for all* $\lambda\in\Lambda$.

Proposition 4.10. *Let the assumptions of Theorem* 4.8 *be satisfied and let* F *be given by* (4.7)*. Then the integrals*

$$\int_0^t \mathrm{D}F(\lambda,x(\tau),\sigma),\qquad \int_0^t f(\lambda,x(s),s)\,\mathrm{d}s\quad\text{and}\quad\int_0^t g(x(s),s)\,\mathrm{d}u(s)$$

exist and the equality

$$\int_0^t \mathrm{D}F(\lambda,x(\tau),\sigma)=\int_0^t f(\lambda,x(s),s)\,\mathrm{d}s+\int_0^t g(x(s),s)\,\mathrm{d}u(s)$$

holds for all $t\in[0,T]$, $\lambda\in\Lambda$ *and* $x\in G([0,T];\mathbb{R}^n)$ *such that* $x(t)\in\Omega$ *for all* $t\in[0,T]$.

The correspondence between solutions of distributional differential equations and generalized ordinary differential equations is clarified by the following theorem. The proof follows easily from Proposition 4.7 and [42], Theorem 4B.1, cf. also [43], Theorem 5.17.

Theorem 4.11. *Let the assumptions of Proposition* 4.10 *be satisfied. Then the couple* $(x,\lambda)\in G([0,T];\mathbb{R}^n)\times\Lambda$ *is a solution of measure differential equation* (4.1) *if and only if it is a solution of the generalized ordinary differential equation* (1.2)*.*

Let us turn our attention back to the periodic problem for the measure differential equation

$$\mathrm{D}x = f(\lambda, x, t) + g(x, t)\mathrm{D}u, \quad x(0) = x(T) \tag{5.1}$$

and look for the sufficient and/or necessary conditions for the existence of points of bifurcation for this problem.

As in Section 4, we will assume that conditions (4.2), (4.5) and (4.6) hold and $F\colon \Lambda\times\Omega\times [0,T]$ is given by (4.7). Then by Proposition 4.9, there are a nondecreasing function $h\colon [0,T] \to \mathbb{R}$ left-continuous on $(0,T]$ and a continuous, increasing function $\omega\colon [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$ and such that $F(\lambda,\cdot,\cdot) \in \mathcal{F}(\Omega \times [0,T], h, \omega)$ for all $\lambda \in \Lambda$. As a result, F satisfies condition (3.2) and, according to Theorem 4.11, problems (5.1) and

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(\lambda, x, t), \quad x(0) = x(T), \tag{5.2}$$

are equivalent.

Furthermore, we will assume also

$$\begin{cases} (x_0,\lambda) \in G([0,T];\mathbb{R}^n) \times \Lambda \text{ is a solution of (5.1) for any } \lambda\in\Lambda \text{ and} \\ \text{there is a } \varrho > 0 \text{ such that } x(t) \in \Omega \text{ for all } t \in [0,T] \text{ and } x \in B(x_0,\varrho). \end{cases} \tag{5.3}$$

Of course, then (3.3) is true, as well.

Analogously to Φ, we define

$$\widetilde{\Phi}(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,\mathrm{d}s + \int_0^t g(x(s), s)\,\mathrm{d}u(s) \tag{5.4}$$
$$\text{for } \lambda\in\lambda,\ x\in B(x_0,\varrho),\ t\in [0,T].$$

By Proposition 4.9, we have

$$\widetilde{\Phi}(\lambda, x)(t) = x(T) + \int_0^t \mathrm{D}F(\lambda, x(\tau), \sigma) = \Phi(\lambda, x)(t) \tag{5.5}$$
$$\text{for } t\in [0,T],\ \lambda\in\Lambda \text{ and } x\in B(x_0,\varrho)$$

and the following statement obviously holds.

Proposition 5.1. *Let the assumptions of Theorem 4.8 be satisfied and let F be given by (4.7). In addition, assume (5.3) and let the operator $\widetilde{\Phi}$ be defined by (5.4). Then $\widetilde{\Phi}(\lambda,\cdot)$ maps $B(x_0,\varrho)$ into $G([0,T];\mathbb{R}^n)$ for any $\lambda\in\Lambda$. Moreover, problem (5.1) is equivalent to finding couples (x,λ) such that $x = \widetilde{\Phi}(\lambda, x)$, as well as to finding solutions (x,λ) of (3.5).*

Thus, it is natural to consider the bifurcation points of the periodic problem (5.1) in the sense of Definition 3.3.

Definition 5.2. Let (5.3) hold. Then the solution $(x_0,\lambda_0)\in G([0,T];\mathbb{R}^n)\times\Lambda$ of (5.1) is said to be a *bifurcation point* of (5.1) if every neighborhood of (x_0,λ_0) in $B(x_0,\varrho)\times\Lambda$ contains a solution (x,λ) of (5.1) such that $x\neq x_0$.

The following statement ensuring the existence of a bifurcation point to the periodic problem (5.1) follows from Theorem 3.5.

Corollary 5.3. *Let the assumptions of Theorem* 4.8 *be satisfied. In addition, assume* (5.3) *and*

$$\tag{5.6} \begin{cases} \text{there is a } \gamma\colon [0,T]\to\mathbb{R} \text{ nondecreasing and such that for any } \varepsilon>0\\ \text{there is } \delta>0 \text{ such that}\\ \left\|\int_s^t [f(\lambda_2,x,r)-f(\lambda_1,x,r)]\,\mathrm{d}r\right\|_n<\varepsilon|\gamma(t)-\gamma(s)|\\ \text{for } x\in\Omega,\ t,s\in[0,T] \text{ and } \lambda_1,\lambda_2\in\Lambda \text{ such that } |\lambda_1-\lambda_2|<\delta. \end{cases}$$

Moreover, let the operator $\widetilde{\Phi}$ be defined by (5.4) *and let $[\lambda_1^*,\lambda_2^*]\subset\Lambda$ be such that*

$$\tag{5.7} x_0 \text{ is an isolated fixed point of the operators } \widetilde{\Phi}(\lambda_1^*,\cdot) \text{ and } \widetilde{\Phi}(\lambda_2^*,\cdot)$$

and

$$\tag{5.8} \mathrm{ind}_{\mathrm{LS}}(\mathrm{Id}-\widetilde{\Phi}(\lambda_1^*,\cdot),x_0)\neq\mathrm{ind}_{\mathrm{LS}}(\mathrm{Id}-\widetilde{\Phi}(\lambda_2^*,\cdot),x_0).$$

Then there is $\lambda_0\in[\lambda_1^*,\lambda_2^*]$ such that (x_0,λ_0) is a bifurcation point of (5.1).

P r o o f. Recall that F is given by (4.7) and hence, by Proposition 5.1, problems (3.1) and (5.1) are then equivalent. Furthermore, we already know that assumptions (3.2) and (3.3) are satisfied. Finally, our assumptions (5.6), (5.7) and (5.8) imply that also all the remaining assumptions of Theorem 3.5 hold. This completes the proof. □

The main goal of this section will be to prove theorems providing the conditions necessary for the pair (λ_0,x_0) to be a bifurcation point for a periodic problem for the measure differential system (5.1). This will be done by Theorems 5.8 and 5.9 related to analogous Theorems 3.11 and 3.13 from Section 3. However, their proofs are not straightforward corollaries of those results for generalized ODEs and several previous steps are needed. The first one consists in finding an explicit formula for the derivative of the function F given by (4.7). This will be given by Proposition 5.5. In its proof we will have to interchange the order of some of the iterated integrals and it will be justified by the following variant of the Bray theorem, cf. [17], Lemma II.17.3.1 and Exercise II.19.3 and [47], Theorem 5.3.13.

Lemma 5.4. *Let* $-\infty < c < d < \infty$, $f \in \mathrm{BV}([0,1];\mathbb{R})$, $h \in G([c,d];\mathbb{R}^n)$. *Let* $K\colon [0,1]\times[c,d] \to \mathcal{L}(\mathbb{R}^n)$ *be such that*

$$(5.9)\qquad \begin{cases} K(\cdot,s) \text{ is regulated for all } s\in[c,d],\\ K(\alpha,\cdot)\in \mathrm{BV}([c,d];\mathcal{L}(\mathbb{R}^n)) \ \text{ for all } \alpha\in[0,1],\\ \text{there is } \varkappa\in(0,\infty) \text{ such that } \|K(\alpha,\cdot)\|_{\mathrm{BV}}\leqslant\varkappa \text{ for all } \alpha\in[0,1]. \end{cases}$$

Then both the iterated integrals

$$(5.10)\qquad \int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha,s)\,\mathrm{d}h(s)\Big) \quad \text{and} \quad \int_c^d\Big(\int_0^1 \mathrm{d}f(\alpha)K(\alpha,s)\Big)\,\mathrm{d}h(s)$$

exist and the equality

$$(5.11)\qquad \int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha,s)\,\mathrm{d}h(s)\Big) = \int_c^d\Big(\int_0^1 \mathrm{d}f(\alpha)K(\alpha,s)\Big)\,\mathrm{d}h(s)$$

holds.

Proof. We can restrict ourselves to the case $n=1$. The extension to a general case is obvious. Let the functions f, K, h satisfy the assumptions of the lemma. By [32], Theorem 6.3.11, the integrals $\int_0^1 \mathrm{d}f(\alpha)K(\alpha,s)$ and $\int_c^d K(\alpha,\sigma)\,\mathrm{d}h(\sigma)$ exist for all $\alpha\in[0,1]$ and $s\in[c,d]$, respectively. Hence, we can define

$$F(s) := \int_0^1 \mathrm{d}f(\tau)K(\tau,s) \quad \text{for } s\in[c,d]$$

and

$$H(\alpha) := \int_c^d K(\alpha,\sigma)\,\mathrm{d}h(\sigma) \quad \text{for } \alpha\in[0,1].$$

To show that F has a bounded variation on $[c,d]$, let us consider an arbitrary division $\{s_0,\dots,s_m\}$ of $[c,d]$ and an arbitrary set of real numbers $\{\xi_i\}_{i=1}^m$ such that $|\xi_i|\leqslant 1$ for all $i\in\{1,\dots,m\}$. Having in mind our assumption (5.9) and Theorem 6.3.6 from [32], we get

$$\begin{aligned}\Big|\sum_{i=1}^m [F(s_i)-F(s_{i-1})]\xi_i\Big| &= \Big|\int_0^1 \mathrm{d}f(\alpha)\Big(\sum_{i=1}^m (K(\alpha,s_i)-K(\alpha,s_{i-1}))\xi_i\Big|\\ &\leqslant \Big(\sup_{\substack{\alpha\in[0,1]\\|\xi_i|\leqslant1}}\Big|\sum_{i=1}^m (K(\alpha,s_i)-K(\alpha,s_{i-1}))\xi_i\Big|\Big)\mathrm{var}_0^1 f\\ &\leqslant \Big(\sup_{\substack{\alpha\in[0,1]\\|\xi_i|\leqslant1}}\Big(\sum_{i=1}^m |K(\alpha,s_i)-K(\alpha,s_{i-1})||\xi_i|\Big)\Big)\mathrm{var}_0^1 f\\ &\leqslant \Big(\sup_{\alpha\in[0,1]} \mathrm{var}_c^d K(\alpha,\cdot)\Big)\mathrm{var}_0^1 f = \varkappa\mathrm{var}_0^1 f < \infty.\end{aligned}$$

In particular, if we put $\xi_i = \operatorname{sign}[F(s_i) - F(s_{i-1})]$ for $i \in \{1, \dots, m\}$ we obtain that the inequality

$$\sum_{i=1}^{m} |F(s_i) - F(s_{i-1})| \leqslant 2\|f\|_{\mathrm{BV}}\, \varkappa < \infty$$

is true for any division $D = \{s_0, \dots, s_m\}$ of $[c, d]$, i.e.,

$$\operatorname{var}_c^d F \leqslant 2\|f\|_{\mathrm{BV}} \varkappa < \infty.$$

Now, making use of Theorem 6.3.11 in [32] once more, we conclude that the integral

$$\int_c^d F(s)\,\mathrm{d}h(s) = \int_c^d \Big(\int_0^1 \mathrm{d}f(\alpha) K(\alpha, s)\Big)\,\mathrm{d}h(s)$$

exists. Further, it is easy to verify that the equalities

$$\int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha, s)\,\mathrm{d}\chi_{[\tau,d]}(s)\Big) = \int_c^d (\mathrm{d}f(\alpha)K(\alpha, s))\,\mathrm{d}\chi_{[\tau,d]}(s) = \int_0^1 \mathrm{d}f(\alpha)K(\alpha, \tau)$$

and

$$\int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha, s)\,\mathrm{d}\chi_{(\sigma,d]}(s)\Big) = \int_c^d (\mathrm{d}f(\alpha)K(\alpha, s))\,\mathrm{d}\chi_{(\sigma,d]}(s) = \int_0^1 \mathrm{d}f(\alpha)K(\alpha, \sigma)$$

hold for all $\tau \in [c, d]$ and $\sigma \in [c, d)$, respectively. Thus, as any finite step function on $[c, d]$ is a finite linear combination of the functions from the set $\{\chi_{[\tau,d]}, \chi_{(\sigma,d]} \colon \tau \in [c, d],\ \sigma \in [c, d)\}$, we can conclude that equality (5.11) is true for each finite step function $h \in G([c, d]; \mathbb{R})$.

Now, let $h \in G([c, d]; \mathbb{R})$ be given and let us choose a sequence of finite step functions $\{h_n\}$ such that $\|h_n - h\|_\infty < n^{-1}$ for each $n \in \mathbb{N}$ and, since F has a bounded variation, by [32], Theorem 6.3.8 (i) we get

$$\begin{aligned}\int_c^d F(s)\,\mathrm{d}h(s) &= \lim_{n\to\infty} \int_c^d F(s)\,\mathrm{d}h_n(s) = \lim_{n\to\infty} \int_c^d \Big(\int_0^1 \mathrm{d}f(\alpha)K(\alpha, s)\Big)\,\mathrm{d}h_n(s)\\ &= \lim_{n\to\infty} \int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha, s)\,\mathrm{d}h_n(s)\Big).\end{aligned}$$

Furthermore, let us put

$$H_n(\alpha) = \int_c^d K(\alpha, s)\,\mathrm{d}h_n(s) \quad \text{for } \alpha \in [0, 1] \text{ and } n \in \mathbb{N}.$$

Then the sequence $\{H_n\}$ is uniformly bounded. Indeed, by (5.9) and [32], Theorem 6.3.7 we have

$$|H_n(\alpha)| \leqslant 2\|K(\alpha, \cdot)\|_{\mathrm{BV}} \|h_n\|_\infty < 2\varkappa(\|h\|_\infty + 1) < \infty \quad \text{for all } \alpha \in [0, 1] \text{ and } n \in \mathbb{N}.$$

Hence, the relations

$$\lim_{n\to\infty}\int_0^1 \mathrm{d}f(\alpha)\Big(\int_c^d K(\alpha,s)\,\mathrm{d}h_n(s)\Big) = \lim_{n\to\infty}\int_0^1 \mathrm{d}f(\alpha)H_n(\alpha) = \int_0^1 \mathrm{d}f(\alpha)H(\alpha)$$

hold by Bounded Convergence Theorem (cf. [32], Theorem 6.3.8) and we can summarize that equality (5.11) is true for each $h \in G([c,d];\mathbb{R})$. This completes the proof. $\square$

In what follows the symbols $f'_x(\lambda,x,t)$ and $g'_x(x,t)$ stand for real $n\times n$-matrices representing, respectively, the total differentials of the functions f and g with respect to x at the points (λ,x,t) or (x,t), respectively, whenever they have a sense.

Proposition 5.5. *Let the assumptions of Theorem* 4.8 *be satisfied and let* F *be given by* (4.7). *Moreover, let the following conditions hold:*

$$(5.12)\quad \begin{cases} \text{for every } (\lambda,x,t)\in\Lambda\times\Omega\times[0,T] \text{ the function } f \text{ has a total differential} \\ f'_x \text{ continuous with respect to } x\in\Omega \text{ for each } \lambda\in\Lambda \text{ and } t\in[0,T] \text{ and} \\ \text{there is a Lebesgue integrable function } \Theta \text{ such that} \\ \|f'_x(\lambda,x,t)\|\leqslant\Theta(t) \text{ for } (\lambda,x,t)\in\Lambda\times\Omega\times[0,T] \end{cases}$$

and

$$(5.13)\quad \begin{cases} \text{for every } (x,t)\in\Omega\times[0,T] \text{ the function } g \text{ has a total differential} \\ g'_x \text{ continuous with respect to } x\in\Omega \text{ for each } t\in[0,T] \text{ and} \\ \text{there is } \varkappa\in(0,\infty) \text{ such that } \|g'_x(x,\cdot)\|_{\mathrm{BV}}\leqslant\varkappa \text{ for all } x\in\Omega. \end{cases}$$

Then for every $(\lambda,x,t)\in\Lambda\times\Omega\times[0,T]$ *the function* F *has a total differential* $F'_x(\lambda,x,t)$ *and it is given by*

$$(5.14)\quad F'_x(\lambda,x,t) = \int_0^t f'_x(\lambda,x,s)\,\mathrm{d}s + \int_0^t g'_x(x,s)\,\mathrm{d}u(s) \quad \text{for all } (\lambda,x,t)\in\Lambda\times\Omega\times[0,T].$$

Moreover, $F'_x(\lambda,\cdot,t)$ *is continuous with respect to* $x\in\Omega$ *for any* $(\lambda,t)\in\Lambda\times[0,T]$.

P r o o f. For $(\lambda,x,t)\in\Lambda\times\Omega\times[0,T]$ denote

$$F_1(\lambda,x,t) = \int_0^t f(\lambda,x,s)\,\mathrm{d}s \quad\text{and}\quad F_2(x,t) = \int_0^t g(x,s)\,\mathrm{d}u(s).$$

Then $F(\lambda,x,t) = F_1(\lambda,x,t) + F_2(x,t)$. By the classical Leibniz Integral Rule (cf. e.g. [29], V.39.1) we have

$$F'_{1,x}(\lambda,x,t) = \int_0^t f'_x(\lambda,x,s)\,\mathrm{d}s \quad\text{for } (\lambda,x,t)\times\Lambda\times\Omega\times[0,T].$$

Analogously, the equality

$$F'_{2,x}(x,t) = \int_0^t g'_x(x,s)\,\mathrm{d}u(s) \quad \text{for } (x,t) \in \Omega \times [0,T] \tag{5.15}$$

could be essentially justified by the measure theory version of the Leibniz Integral Rule, cf. e.g. [48], Proposition 23.37. However, our setting is little bit different. Hence, we feel that it would be honest to give here an independent proof. Let $(z,x,t) \in \mathbb{R}^n \times \Omega \times [0,T]$ be given, while $x + z \in \Omega$. By the Mean Value Theorem (cf. [20], Lemma 8.11), we have

$$\begin{aligned}\frac{F_2(x+\theta\, z,t) - F_2(x,t)}{\theta} &= \int_0^t \left[\frac{g(x+\theta z,s) - g(x,s)}{\theta}\right] \mathrm{d}u(s) \\ &= \left(\int_0^t \left(\int_0^1 [g'_x(\alpha(x+\theta z) + (1-\alpha)x,s)]\,\mathrm{d}\alpha\right) \mathrm{d}u(s)\right) z\end{aligned}$$

for any $\theta > 0$ sufficiently small. Obviously, the functions

$$f(\alpha) := \alpha, \quad K_\theta(\alpha,s) := g'_x(\alpha(x+\theta z) + (1-\alpha)x,s) \quad \text{and} \quad u := h$$

satisfy the assumptions of Lemma 5.4. Hence, by Lemma 5.4 we have

$$\begin{aligned}&\left(\int_0^t \left(\int_0^1 [g'_x(\alpha(x+\theta z) + (1-\alpha)x,s)]\,\mathrm{d}\alpha\right) \mathrm{d}u(s)\right) \\ &\qquad = \left(\int_0^1 \left(\mathrm{d}\alpha \int_0^t [g'_x(\alpha(x+\theta z) + (1-\alpha)\,x,s)]\,\mathrm{d}u(s)\right)\right).\end{aligned}$$

Furthermore,

$$\lim_{\theta\to 0+} g'_x(\alpha(x+\theta z) + (1-\alpha)\,x,s) = g'_x(x,s)$$

and

$$\|g'_x(\alpha(x+\theta z) + (1-\alpha)x,s))\| \leqslant \varkappa < \infty \quad \text{for all } (\alpha,s) \in [0,1] \times [0,T].$$

Therefore, by Bounded Convergence Theorem (see [32], Theorem 6.3.8) we obtain

$$\lim_{\theta\to 0+} \int_0^t g'_x(\alpha(x+\theta z) + (1-\alpha)x,s)\,\mathrm{d}u(s) = \int_0^t g'_x(x,s)\,\mathrm{d}u(s).$$

This proves (5.14). The continuity of $F'_x(\lambda,\cdot,t)$ follows from the continuity assumptions contained in (5.12) and (5.13). □

Proposition 4.10 can be easily modified to a matrix valued function. Therefore, we can state the following assertion.

Proposition 5.6. *Let the assumptions of Proposition* 5.5 *be satisfied. Then all the integrals*

$$\int_0^t \mathrm{D}F'_x(\lambda, x(\tau), \sigma), \qquad \int_0^t f'_x(\lambda, x(s), s)\,\mathrm{d}s, \qquad \int_0^t g'_x(x(s), s)\,\mathrm{d}u(s)$$

exist and the equality

$$\int_0^t \mathrm{D}[F'_x(\lambda, x(\tau), \sigma)] = \int_0^t f'_x(\lambda, x(s), s)\,\mathrm{d}s + \int_0^t g'_x(x(s), s)\,\mathrm{d}u(s) \tag{5.16}$$

holds for all $t \in [0, T]$, $\lambda \in \Lambda$ *and* $x \in G([0, T]; \mathbb{R}^n)$ *such that* $x(t) \in \Omega$ *for all* $t \in [0, T]$.

Next result characterizes the derivative $\widetilde{\Phi'}_x$ of the operator $\widetilde{\Phi}$.

Proposition 5.7. *Let* $\widetilde{\Phi}$ *be given by* (5.4) *and let the assumptions of Proposition* 5.5 *be satisfied. Then for given* $(\lambda, x) \in \Lambda \times B(x_0, \varrho)$, *the derivative* $\widetilde{\Phi'}_x(\lambda, x)$ *of* $\widetilde{\Phi}(\lambda, \cdot)$ *at* x *is given by*

$$(\widetilde{\Phi}'_x(\lambda, x)z)(t) = z(T) + \int_0^t f'_x(\lambda, x(s), s)z(s)\,\mathrm{d}\tau + \int_0^t g'_x(x(s), s)z(s)\,\mathrm{d}u(s) \tag{5.17}$$

for all $z \in G([0, T]; \mathbb{R}^n)$ *and* $t \in [0, T]$.

P r o o f. Recall that by (5.5) we have $\widetilde{\Phi}(\lambda, x)(t) = \Phi(\lambda, x)(t)$ and therefore, also

$$(\widetilde{\Phi}'_x(\lambda, x)z)(t) = (\Phi'_x(\lambda, x)z)(t)$$

for $(\lambda, x, t) \in \Lambda \times B(x_0, \varrho) \times [0, T]$ and $z \in G([0, T]; \mathbb{R}^n)$. Thus, since by (3.11) from Proposition 3.8 we have

$$(\Phi'_x(\lambda, x)\,z)(t) = z(T) + \int_0^t \mathrm{D}[F'_x(\lambda, x(\tau), \sigma)z(\tau)]$$

for $(\lambda, x, t) \in \Lambda \times B(x_0, \varrho) \times [0, T]$ and $z \in G([0, T]; \mathbb{R}^n)$, it is enough to show that the relation

$$\int_0^t D[F'_x(\lambda, x(\tau), \sigma)z(\tau)] = \int_0^t f'_x(\lambda, x(\tau), \tau)z(\tau)\,\mathrm{d}\tau + \int_0^t g'_x(x(\tau), \tau), z(\tau)\,\mathrm{d}u(\tau) \tag{5.18}$$

holds for $(\lambda, x, t) \in \Lambda \times B(x_0, \varrho) \times [0,T]$ and $z \in G([0,T];\mathbb{R}^n)$. This will be done in a way analogous to that used in the proof of item 2 of Lemma 5.1 in [44], only we should use Proposition 5.6 instead of Proposition 5.12 from [43]. By Proposition 5.6, relation (5.16) is true for every $x \in G([0,T];\mathbb{R}^n)$. Furthermore, if $[\alpha,\beta] \subset [0,T]$, $z \in G([0,T];\mathbb{R}^n)$ and $z(t) = \widetilde{z} \in \mathbb{R}^n$ for $t \in (\alpha,\beta)$, then by (5.16), Lemma 2.3 and Hake Theorem (cf. e.g. [32], Theorem 6.5.6) we deduce

$$\begin{aligned}
&\int_\alpha^\beta \mathrm{D}[F'_x(\lambda, x(\tau), \sigma) z(\tau)] \\
&= \lim_{\delta\to 0+}\Bigg(\int_{\alpha+\delta}^{\beta-\delta} \mathrm{D}[F'_x(\lambda, x(\tau), \sigma)]\widetilde{z} + \int_\alpha^{\alpha+\delta} \mathrm{D}[F'_x(\lambda, x(\tau), \sigma) z(\tau)] \\
&\quad + \int_{\beta-\delta}^{\beta} \mathrm{D}[F'_x(\lambda, x(\tau), \sigma) z(\tau)]\Bigg) \\
&= \int_\alpha^\beta f'_x(\lambda, x(\tau), \tau) z(\tau)\,\mathrm{d}\tau \\
&\quad + \lim_{\delta\to 0+}\Bigg(\int_{\alpha+\delta}^{\beta-\delta} g'_x(x(\tau),\tau) z(\tau)\,\mathrm{d}u(\tau) + (F'_x(\lambda, x(\alpha), \alpha+\delta) \\
&\quad - F'_x(\lambda, x(\alpha), \alpha)) z(\alpha) + (F'_x(\lambda, x(\beta), \beta) - F'_x(\lambda, x(\beta), \beta-\delta)) z(\beta)\Bigg) \\
&= \int_\alpha^\beta f'_x(\lambda, x(\tau), \tau)\, z(\tau)\,\mathrm{d}\tau \\
&\quad + \lim_{\delta\to 0+}\Bigg(\int_{\alpha+\delta}^{\beta-\delta} g'_x(x(\tau),\tau) z(\tau)\,\mathrm{d}u(\tau) + g'_x(x(\alpha),\alpha) z(\alpha)(h(\alpha+\delta) - h(\alpha)) \\
&\quad + g'_x(x(\beta),\beta) z(\beta)(h(\beta) - h(\beta-\delta))\Bigg) \\
&= \int_\alpha^\beta f'_x(\lambda, x(\tau), \tau) z(\tau)\,\mathrm{d}\tau + \int_\alpha^\beta g'_x(x(\tau),\tau) z(\tau)\,\mathrm{d}u(\tau).
\end{aligned}$$

Having in mind that every regulated function is a uniform limit of finite step functions, we complete the proof by means of the Uniform Convergence Theorem, cf. e.g. [32], Theorem 6.8.2.

□

We are now able to establish the conditions necessary for the given couple (λ_0, x_0) to be a bifurcation point for problem (5.1). This will be the content of the following theorem.

Theorem 5.8. *Let the assumptions of Proposition* 5.5 *be satisfied. Moreover, assume that* (5.3) *and* (5.6) *hold and*

$$
(5.19)\quad \begin{cases} \textit{there is a nondecreasing function } \widetilde{\gamma}\colon\ [0,T]\to\mathbb{R} \textit{ such that for any } \varepsilon>0 \\ \textit{there is a } \delta>0 \textit{ such that} \\ \left\| \int_s^t [f'_x(\lambda_1,x,r)-f'_x(\lambda_2,y,r)]\,\mathrm{d}r + \int_s^t [g'_x(x,r)-g'_x(y,r)]\,\mathrm{d}u(r) \right\|_{n\times n} \\ <\varepsilon|\widetilde{\gamma}(t)-\widetilde{\gamma}(s)| \textit{ for all } t,s\in\ [0,T] \textit{ and all } x,y\in\Omega,\ \lambda_1,\lambda_2\in\Lambda \\ \textit{satisfying } |\lambda_1-\lambda_2|+\|x-y\|_n<\delta. \end{cases}
$$

Let the operator $\widetilde{\Phi}$ be defined by (5.4) *and let $\lambda_0\in\Lambda$ be given. Let* $\mathrm{Id}-\widetilde{\Phi'}_x(\lambda_0,x_0)$ *be an isomorphism of $G([0,T];\mathbb{R}^n)$ onto $G([0,T];\mathbb{R}^n)$. Then there is $\delta>0$ such that (x,λ) is not a bifurcation point of the equation $\widetilde{\Phi}(\lambda,x)=x$ whenever $\|x-x_0\|_\infty+|\lambda-\lambda_0|<\delta$.*

P r o o f. Recall that in addition to (5.3), (5.6) and (5.19), we assume, similarly as in Proposition 5.5, that conditions (4.2), (4.5), (4.6), (5.12) and (5.13) hold, as well. Let F be given by (4.7). Then by Proposition 5.5, its derivative with respect to x is given by (5.14), i.e.,

$$
F'_x(\lambda,x,t)=\int_0^t f'_x(\lambda,x,s)\,\mathrm{d}s+\int_0^t g'_x(x,s)\,\mathrm{d}u(s)\quad \text{for all } (\lambda,x,t)\in\Lambda\times\Omega\times\ [0,T].
$$

Furthermore, by Proposition 5.7, the derivative with respect to x of $\widetilde{\Phi}(\lambda,\cdot)$ is given by (5.17), i.e.,

$$
\widetilde{\Phi}'_x(\lambda,x)z(t)=z(T)+\int_0^t f'_x(\lambda,x(s),s)z(s)\,\mathrm{d}\tau+\int_0^t g'_x(x(s),s)z(s)\,\mathrm{d}u(s)
$$

for all $(\lambda,x)\in\Lambda\times B(x_0,\varrho)$, $z\in G([0,T];\mathbb{R}^n)$ and $t\in[0,T]$. Moreover, by relation (5.18), from the proof of the same proposition we have

$$
(5.20)\qquad (\widetilde{\Phi'}_x(\lambda,x)z)(t)=(\Phi'_x(\lambda,x)z)(t)\quad \text{for } (\lambda,x)\in\Lambda\times B(x_0,\varrho),\ z\in G([0,T];\mathbb{R}^n) \text{ and } t\in[0,T],
$$

where Φ and Φ'_x are, respectively, given by (3.4) and (3.11).

Now, suppose that $\mathrm{Id}-\widetilde{\Phi}'_x(\lambda_0,x_0)\colon G([0,T];\mathbb{R}^n)\to G([0,T];\mathbb{R}^n)$ is an isomorphism. Then, due to (5.20), the mapping $\mathrm{Id}-\Phi'_x(\lambda_0,x_0)\colon G([0,T];\mathbb{R}^n)\to G([0,T];\mathbb{R}^n)$ is an isomorphism, as well.

We want to apply Theorem 3.11. To this aim we need to verify that all its assumptions, i.e., (3.2), (3.3), (3.7), (3.10) and (3.20), are satisfied.

First, notice that the periodic problem for equation (4.1) is by Theorem 4.8 equivalent to the periodic problem for the integral equation (4.3). Furthermore, by Proposition 4.9 there are a nondecreasing function $h\colon [0,T]\to\mathbb{R}$ left-continuous on $(0,T]$ and a continuous, increasing function $\omega\colon [0,\infty)\to\mathbb{R}$ with $\omega(0)=0$ and such that $F(\lambda,\cdot,\cdot)\in\mathcal{F}(\Omega\times[0,T],h,\omega)$ for all $\lambda\in\Lambda$. In particular, (3.2) is satisfied. Moreover, Theorem 4.11 implies that the periodic problem (3.1) is equivalent with the periodic problem for equation (4.3) and hence, F satisfies also (3.3).

Second, from (5.12), (5.13) and (5.14) it follows immediately that (3.12) is also true if we put

$$\widetilde{h}(t)=\int_0^t \Theta(r)\,\mathrm{d}r+\varkappa\,\mathrm{var}_0^t u].$$

Finally, it remains to show that (3.13) is satisfied, too. By (5.14) and (5.19) there is a nondecreasing function $\widetilde{\gamma}\colon [0,T]\to\mathbb{R}$ such that for any $\varepsilon>0$ there is a $\delta>0$:

$$\begin{aligned}\|F'_x(\lambda_1,x,t)-F'_x(\lambda_2,y,t)-F'_x(\lambda_1,x,s)+F'_x(\lambda_2,y,s)\|_{n\times n}\\ =\left\|\int_s^t [f'_x(\lambda_1,x,r)-f'_x(\lambda_2,y,r)]\,\mathrm{d}r+\int_s^t [g'_x(x,r)-g'_x(y,r)]\,\mathrm{d}u(r)\right\|_{n\times n}\\ <\varepsilon|\widetilde{\gamma}(t)-\widetilde{\gamma}(s)|\end{aligned}$$

for all $t,s\in[0,T]$ and all $x,y\in\Omega$, $\lambda_1,\lambda_2\in\Lambda$ such that $|\lambda_1-\lambda_2|+\|x-y\|_n<\delta$. This means that (3.13) and (3.20) are true when we take $\lambda_1=\lambda_2$ and $x=y$ in the last inequality. Moreover, by (5.6), we obtain that also (3.7) is satisfied. Thus, all the hypotheses of Theorem 3.11 are satisfied. Therefore, (λ_0,x_0) is not a bifurcation point of the equation $\widetilde{\Phi}(\lambda,x)=x$ and there is $\delta>0$ such that (x,λ) is not a bifurcation point of this equation whenever $\|x-x_0\|_\infty+|\lambda-\lambda_0|<\delta$. This completes the proof. □

Finally, analogously to Theorem 3.13 we can state a necessary condition for the existence of the bifurcation point to problem (5.1) in the form related to the Fredholm type alternative.

Theorem 5.9. *Let the assumptions of Theorem* 5.8 *be satisfied and let the couple* $(\lambda_0,x_0)\in\Lambda\times\Omega$ *be a bifurcation point of problem* (5.1). *Then there exists* $q\in G([0,T];\mathbb{R}^n)$ *such that the equation*

$$z(t)-z(T)-\int_0^t f'_x(\lambda_0,x_0(\tau),\tau)z(\tau)\,\mathrm{d}\tau-\int_0^t g'_x(x_0(\tau),\tau)z(\tau)\,\mathrm{d}u(\tau)=q(t)$$

for $t\in[0,T]$ *has no solution in* $G([0,T];\mathbb{R}^n)$ *and the corresponding homogeneous equation*

$$z(t)-z(T)-\int_0^t f'_x(\lambda_0,x_0(\tau),\tau))z(\tau)\,\mathrm{d}\tau-\int_0^t g'_x(x_0(\tau),\tau)z(\tau)\,\mathrm{d}u(\tau)=0$$

for $t\in[0,T]$ *possesses at least one nontrivial solution in* $G([0,T];\mathbb{R}^n)$.

P r o o f. Suppose (λ_0, x_0) is a bifurcation point of (5.1), i.e., of the equation

$$\widetilde{\Phi}(\lambda, x) = x$$

with $\widetilde{\Phi}$ given by (5.4). Then by Proposition 4.9, (λ_0, x_0) is also a bifurcation point of the equation $\Phi(\lambda, x) = x$, where Φ is given by (3.4). Our statement follows by Theorem 3.13. □

Next example has been already considered in [12], where conditions ensuring the existence of a bifurcation point for the impulsive problem (5.21) were stated. Due to our Theorem 5.9, we are now able to show also its uniqueness.

Example 5.10. Consider the impulsive problem

$$x' = \lambda b(t)x + c(t)x^2, \quad \Delta^+ x\Big(\frac{1}{2}\Big) = x^2\Big(\frac{1}{2}\Big), \quad x(0) = x(1) \tag{5.21}$$

with $b, c \in L^1[0,1]$ and $\int_0^1 b(s)\,\mathrm{d}s \neq 0$, i.e.,

$$x(t) = x(1) + \int_0^t f(\lambda, x(s), s)\,\mathrm{d}s + \int_0^t g(x(s), s)\,\mathrm{d}u(s),$$

where $f(\lambda, x, s) = \lambda\, b(s)x + c(s)x^2$, $g(x, s) = x^2$, $u(s) = \chi_{(1/2,1]}(s)$.

Obviously, $x_0(t) \equiv 0$ is a solution of (5.21) for all λ. Linearization at x_0 yields

$$z' = \lambda b(t)z, \quad z(0) = z(1) \Leftrightarrow \begin{cases} \lambda = 0 \wedge z \equiv \text{const}, \\ \lambda \neq 0 \wedge z \equiv 0. \end{cases} \tag{5.22}$$

It was shown in Example 6.12 of [12] that the assumptions of Corollary 5.3 are satisfied with $\lambda_2^* = -\lambda_1^* > 0$ sufficiently small. Thus, by Corollary 5.3 it follows that for any $\delta > 0$ there is $\lambda^* \in (-\delta, \delta)$ such that $(\lambda^*, 0)$ is a bifurcation point of (5.21). It is worth noticing that Corollary 5.3 does not ensure that it has to be $\lambda^* = 0$. In fact, it could happen that there is a line segment $J = (-\widetilde{\delta}, \widetilde{\delta})$ such that any couple $(\lambda, 0)$, with $\lambda \in J$ is a bifurcation point of (5.21).

On the other hand, we can verify that f, g, h fulfil the assumptions of Theorem 5.9. Thus, in view of (5.22) and Theorem 5.9, we conclude that $(\lambda, 0)$ cannot be a bifurcation point of (5.12) whenever $\lambda \neq 0$. Consequently, $(0, 0)$ is the only bifurcation point of (5.21).

For further example the following special case of the result by Lomtatidze (cf. [27], Theorem 11.1 and Remark 0.5) will be useful.

Proposition 5.11. *Let $q\colon [0,T] \to \mathbb{R}$ be continuous and such that*

$$\int_0^T q_-(s)\,\mathrm{d}s > 0 \quad \text{and} \quad \int_0^T q_+(s)\,\mathrm{d}s > 0$$

where, as usual,

$$q_+(t) := \max\{q(t), 0\} \quad \text{and} \quad q_-(t) := -\min\{q(t), 0\} \quad \text{for } t \in [0,T].$$

Further, assume that

$$\int_0^T q_-(s)\,\mathrm{d}s < \Big(1 - \frac{\pi}{2}\int_0^T q_-(s)\,\mathrm{d}s\Big)\Big(\int_0^T q_+(s)\,\mathrm{d}s\Big) \text{ and } \int_0^T q_-(s)\,\mathrm{d}s < \frac{2}{\pi}. \tag{5.23}$$

Then the equation $y'' + q(t)\,y = 0$ has only trivial T-periodic solution.

Example 5.12. By Example (4.2) in [8] (cf. also [9], Remark 3.1) the function

$$y(t) = y_0(t) = (2 + \cos t)^3$$

is a solution of the problem

$$y''(t) = (6.6 - 5.7\cos t - 9\cos^2 t)y^{1/3} - 0.3y^{2/3}, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\,\pi)$$

related to the Liebau valveless pumping phenomena. Since $y_0(\pi) = 1$,

$$2(y_0(\pi))^3 - (y_0(\pi))^2 - 4y_0(\pi) + 3 = 0$$

and

$$(2 + \cos t)y_0'(t) + 3(\sin t)y_0(t) = 0 \quad \text{for all } t \in [0, 2\pi],$$

$y = y_0$ clearly solves also the parameterized impulsive problem

$$\begin{aligned} &y'' = \lambda((2 + \cos t)y' + 3(\sin t)y) + (6.6 - 5.7\cos t - 9\cos^2 t)y^{1/3} - 0.3y^{2/3}, \\ &\Delta^+ y(\pi) = 2(y(\pi))^3 - (y(\pi))^2 - 4y(\pi) + 3, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi) \end{aligned} \tag{5.24}$$

for all $\lambda \in \mathbb{R}$.

To prove that the couple $(y_0, 0)$ is not a bifurcation point of (5.24), we want to apply Theorem 5.9. To this aim, we rewrite problem (5.24) as the integral system

$$x(t) = x(2\pi) + \int_0^t f(\lambda, x(s), s)\,\mathrm{d}s + \int_0^t g(x(s), s)\,\mathrm{d}u(s), \tag{5.25}$$

where

$$x_1 = y, \quad x_2 = y', \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$f(\lambda, x, t) = \begin{pmatrix} x_2 \\ \lambda((2+\cos t)x_2 + 3(\sin t)x_1) + R(t)x_1^{1/3} - 0.3x_1^{2/3} \end{pmatrix},$$

$$g(x, t) = \begin{pmatrix} 2x_1^3 - x_1^2 - 4x_1 + 3 \\ 0 \end{pmatrix}, \quad u(t) = \chi_{(\pi, 2\pi]}(t)$$

and

$$R(t) = 6.6 - 5.7\cos t - 9\cos^2 t.$$

Obviously, $x_0 = \binom{y_0}{y_0'}$ is a solution to (5.25) for all $\lambda \in \mathbb{R}$. Choose $\Omega = (0.5, 28) \times (-20, 20)$, $\Lambda = (-1, 1)$ and $\varrho = 0.25$. Then it is possible to verify that $x(t) \in \Omega$ for all $t \in [0, 2\pi]$ whenever $x \in B(x_0, \varrho)$ and we can conclude that f, g and h satisfy conditions (4.2) and (5.3). Moreover, it is easy to verify that assumptions (4.5), (4.6), (5.3), (5.12) and (5.13) are satisfied, as well.

Next we show that also (5.19) holds. To this aim, consider the expression

$$\Delta(t, s, v, w, \lambda_1, \lambda_2) := \int_s^t [f_x'(\lambda_1, v, r) - f_x'(\lambda_2, w, r)]\,\mathrm{d}r + \int_s^t [g_x'(v, r) - g_x'(w, r)]\,\mathrm{d}u(r),$$

where $0 \leqslant s < t \leqslant 2\pi$, $v, w \in \Omega$, and $\lambda_1, \lambda_2 \in \Lambda$. As

$$(5.26) \qquad f_x'(\lambda, x, t) = \begin{pmatrix} 0 & 1 \\ \lambda 3\sin t + \frac{1}{3}R(t)x_1^{-2/3} - 0.2x_1^{-1/3} & \lambda(2+\cos\,t) \end{pmatrix},$$

$$g_x'(x, t) = \begin{pmatrix} 6x_1^2 - 2x_1 - 4 & 0 \\ 0 & 0 \end{pmatrix}$$

for $\lambda \in \Lambda$, $x \in \Omega$ and $t \in [0, 2\pi]$, it is not difficult to justify the inequality

$$\begin{aligned} \|\Delta(t, s, v, w, \lambda_1, \lambda_2)\|_{2\times 2} &\leqslant 5(|\lambda_1 - \lambda_2| + |v_1^{-1/3} - w_1^{-1/3}| + |v_1^{-2/3} - w_1^{-2/3}|)(t-s) \\ &\quad + (6|v_1^2 - w_1^2| + 2|v_1 - w_1|)(u(t) - u(s)) \end{aligned}$$

for $0 \leqslant s < t \leqslant 2\pi$, $v, w \in \Omega$, and $\lambda_1, \lambda_2 \in \Lambda$. Now, having in mind that the functions x^2, $x^{-1/3}$ and $x^{-2/3}$ are uniformly continuous on $[0.5, 28]$, it is already easy to verify that assumption (5.19) will be satisfied if we put $\widetilde{\gamma}(t) = t + u(t)$.

Finally, since

$$\left\| \int_s^t [f(\lambda_1, x, r) - f(\lambda_2, x, r)]\,\mathrm{d}r \right\|_2 \leqslant |\lambda_1 - \lambda_2|[3(|x_2| + |x_1|)](t-s)$$

for $0 \leqslant s < t \leqslant 2\pi$, $x \in \Omega$, and $\lambda_1, \lambda_2 \in \Lambda$, we can see that assumption (5.6) will be satisfied with $\gamma(t) = t$.

The linearization of (5.25) around $(x_0, 0)$ is

$$z(t) = z(2\pi) + \int_0^t f'_x(0, x_0(r), r) z(r)\,\mathrm{d}r + g'_x(x_0(\pi), \pi) z(\pi) \chi_{(0,\pi]}(t) \tag{5.27}$$

for $t \in [0, 2\pi]$. Inserting $\lambda = 0$ and $x_0 = \binom{y_0}{y'_0}$ into (5.26), we get

$$\begin{aligned} f'_x(0, x_0(t), t) &= \begin{pmatrix} 0 & 1 \\ \frac{1}{3} R(t)(y_0(t))^{-2/3} - 0.2(y_0(t))^{-1/3} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ \dfrac{3(6 - 7\cos t - 10\cos^2 t)}{10(2 + \cos t)^2} & 0 \end{pmatrix} \end{aligned}$$

and

$$g'_x(x_0(\pi), \pi) = \begin{pmatrix} 6(x_0(\pi))^2 - 2y_0(\pi) - 4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that (5.27) reduces to the second order periodic problem

$$z'' = q(t) z, \quad z(0) = z(2\pi), \quad z'(0) = z'(2\pi), \tag{5.28}$$

where

$$q(t) = \frac{3(6 - 7\cos t - 10\cos^2 t)}{10(2 + \cos t)^2} \quad \text{for } t \in [0, 2\pi].$$

One can compute:

$$\int_0^{2\pi} q_-(s)\,\mathrm{d}s = 2\pi - \frac{5(6 + 59 \arctan 1/3)}{5\sqrt{3}} \approx 0.513\,543 < \frac{2}{\pi} \approx 0.636\,62.$$

In particular,

$$0 < 1 - \frac{\pi}{2} \int_0^{2\pi} q_-(s)\,\mathrm{d}s \approx 0.193\,328.$$

Furthermore,

$$\int_0^{2\pi} q_+(s)\,\mathrm{d}s = \frac{1}{15}((59\sqrt{3} - 60)\pi - 2\sqrt{3}(6 + \arctan 1/3)) \approx 3.066\,82,$$

and

$$\int_0^{2\pi} q_-(s)\,\mathrm{d}s \approx 0.513\,543 < \left(1 - \frac{\pi}{2} \int_0^{2\pi} q_-(s)\,\mathrm{d}s\right)\left(\int_0^{2\pi} q_+(s)\,\mathrm{d}s\right) \approx 0.592\,902.$$

Consequently, Proposition 5.11 implies that the linear problem (5.28) possesses only the trivial solution. Thus, by Theorems 5.8 and 5.9, we conclude that there is a $\delta > 0$ such that (y, λ) is not a bifurcation point of (5.24) whenever $|\lambda| + \|y - y_0\|_\infty < \delta$. In particular, the couple $(y_0, 0)$ can not be a bifurcation point of (5.24).

Note that the validity of the assumptions of Theorems 5.8 and 5.9 for the model worked out in this example can also be verified using Corollary 2.1 in [15].

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Authors' addresses: Maria Carolina Mesquita, Department of Mathematics, Federal University of São Carlos, Caixa Postal 676, 13565.905 São Carlos SP, Brazil, e-mail: `mc12stefani@hotmail.com`, Milan Tvrdý (corresponding author), Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague, Czech Republic, e-mail: `tvrdy@math.cas.cz`.